ARRIVAL TIME IN QUANTUM MECHANICS

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1 Introduction

In this review we will be concerned with the problem of the arrival time in quantum mechanics, where we consider a particle in one dimension, located in $x < 0$ and consisting entirely of positive momenta, and try to find the probability that it crosses $x = 0$ between a time T and $T + dT$.

In general, the question of time in quantum mechanics is one which continues to be the subject of much research and disagreement. From a basic knowledge of quantum mechanics, one might assume that the issue had been resolved, at least in the nonrelativistic case. One of the first equations one encounters in QM is the uncertainty relation

$$\Delta E \Delta t \geq \frac{1}{2}. \tag{1.1}$$

(Throughout this work, we use the convention that $\hbar = 1$.) This is commonly interpreted as expressing the fact that a measurement of a system's energy is inversely proportional to the length of time for which one observes the system. However, throughout the past century there has in fact been a great deal of debate about what such an energy-time uncertainty relation actually means [1]. Now recall that another result from elementary QM is that position-momentum uncertainty can be expressed as [2]

$$(\Delta x)^2(\Delta p)^2 \geq -\frac{1}{4}(\langle[\hat{x},\hat{p}]\rangle)^2. \tag{1.2}$$

Since the $\hat{x}$ and $\hat{p}$ are canonically conjugate variables, then $[\hat{x},\hat{p}] = i$ and we recover the familiar position-momentum uncertainty relation. Given the time-energy uncertainty relation, we may then ask if there is a time operator $\hat{T}$ canonically conjugate to the Hamiltonian which leads to this relation.

To begin with, we must make a very important distinction between *parameters* and *dynamical variables*. In the standard Schrödinger equation, time is merely a parameter, and we are concerned with measuring quantities such as position and momentum at a given time. However, we may also seek to measure the time at which a quantum system is in a certain state, and in this case the relevant time depends upon the evolution of a dynamical system. We may think of the distinction as being similar to the difference between space coordinates as general external parameters, and the measurement of position for a particular system via the operator $\hat{x}$ [3, 4]. This is the context in which we meet the arrival time problem. In this work, as we said, we will be concerned primarily with this problem in one dimension, where we have say a particle at $x < 0$ and ask at what time it will arrive at the position $x = 0$. For the purposes of this review, we will also consider the problem in the absence of any potential which might cause particles to cross the arrival point several times, so that the first crossing is the only crossing. In the same way that position at constant time is given by a probability distribution $P(x) = |\psi(x)|^2 = |\langle x|\psi\rangle|^2$ for a state $|\psi\rangle$, we expect the arrival time to be given by a probability distribution which we denote $\Pi(T)$, so that $\Pi(T)dT$ denotes the probability that the particle crosses the arrival point between T and $T + dT$. Such a distribution should clearly be normalizable as in the case of the position distribution, so that the total probability for detection is 1. The reason this is not as straightforward as in the position case is that, as Pauli argued in the 1930s, it does not seem possible to define a self-adjoint time operator $\hat{T}$ canonically conjugate to $\hat{H}$ due to the semi-bounded nature of the Hamiltonian's spectrum - i.e. there are no negative eigenvalues or corresponding eigenstates of the Hamiltonian [3]. Since standard QM requires observables to be represented by self-adjoint operators with orthogonal eigenstates, the non-existence of such an operator may make it problematic to put arrival time on the same footing as other observables in quantum theory. This issue will be extensively discussed in the following chapters. Regardless of the answer to this question, it seems clear that there ought to be some theoretical means of deriving an arrival time distribution, since it is an experimental fact that the arrival time distribution of a particle at a screen can be measured. This dissertation will present a review of some of the main approaches to this subject over the last couple of decades, although earlier works will also be examined where they are relevant to more recent developments. We will, for example, look at arguments by Allcock and Kijowski from the 1960s and

1970s, because they were highly influential and continue to be cited [5–8].

We shall begin in chapter 2 by examining some useful background before addressing some of the main approaches to the problem. We shall first look at some long-established possible candidates for the arrival time distribution. In analogy with classical mechanics, we might define the arrival time distribution at $x = 0$ to be the equal to the quantum probability current density at that point $J(0, T)$. This is problematic due to its having negative values under certain conditions, which obviously disqualifies it from being a candidate for a probability distribution. Another widely cited possibility is given by Kijowski's distribution [8]. Rather than taking the probability current density, Kijowski writes down axioms from which one can derive the classical version of this quantity a priori. He then posits a quantum version of these axioms and obtains a probability distribution which is distinct from $J(0, T)$. This probability distribution has been cited again and again in the arrival time literature over the years, and we will find that many approaches derive it by alternative means. We shall then briefly discussion the reason for the negative values of $J(0, T)$, the quantum backflow effect, since this has been an active area of recent research which will occasionally be mentioned later on. We will also describe the *Zeno effect*, which is the tendency of a constantly monitored particle to remain in the same state, as it will be important later on.

In chapter 3, we shall examine some of the work done on an arrival time operator and arrival time eigenstates. We will firstly look at Pauli's argument against the possibility of defining a self-adjoint arrival time operator. Despite his influential argument, attempts have nonetheless been made to define such an operator. The most popular candidate is the Aharonov-Bohm operator. We will look at its eigenstates and find that they are not orthogonal, suggesting that this operator cannot correspond to an observable. In an attempt to circumvent this, we will see how Allcock tries to define arrival time eigenstates in the presence of a source, which due to the modified boundary conditions results in the energy spectrum being extended to negative eigenvalues. We will see how he argues that even these eigenstates cannot be orthogonal [5]. Being aware of these difficulties, some authors have tried in more recent times to modify the Aharonov-Bohm operator to obtain one which *is* self-adjoint. We will see how they were successful in obtaining arrival time distributions. We will also examine the argument that the operator does not in fact have to be self-adjoint in order to obtain a physical theory with a plausible arrival time distribution. Finally in chapter 3, we will look at a refutation of Pauli's argument, and an attempt to define a self-adjoint time operator for a particle in a box.

In chapter 4 we will look at how systems with detectors and clocks have been modelled in an attempt to obtain an arrival time distribution. This is in response to the concerns expressed by some authors that the arrival time cannot be obtained from excessive idealizations, but must in fact take account of the fact that any arrival time distribution results from the process of detection. The basic idea of this family of approaches is that the Schrödinger equation is solved with a modified Hamiltonian $\hat{H}_o \rightarrow \hat{H}_o + \hat{H}_{apparatus}$. The idea expressed by many authors is that the resulting distributions are convolutions of the *ideal* (apparatus-independent) distribution with response functions depending on the particular apparatus. One may, then, in principle, obtain the ideal distribution by deconvolution. We will see how this has been done for different models with varying success.

Finally, in chapter 5, we will look at attempts to address the arrival time problem in the context of the decoherent histories approach. This basic idea of this approach to QM is that in a closed system we may consider all possible histories of the system, and attempt to assign probabilities to each different alternative. However, between the different histories there is usually quantum interference which prevents the probabilities of histories from being disjoint and additive, something which we clearly require in order to define appropriate probabilities for physical theories. When we impose the condition of non-interference, which may be obtained by grouping together sets of histories, we can often obtain suitable probability distributions. Many authors have attempted to apply this to the arrival time problem. We shall see how Yamada and Takagi were among the first to apply it to this problem, reaching a negative conclusion about the possibility of defining an arrival time distribution. However, we shall look at more recent work which contests this conclusion, on the basis that it failed to take account of the Zeno effect, and which overcomes this issue to succeed in obtaining a plausible arrival time distribution.

2 Background

In this chapter we will look at some work on arrival time which is important for an understanding of much of the work done on the subject in recent years. We will begin by looking at the classical expression for arrival time distribution, and examine how it can be quantized to give an expression which recurs time and time again in different approaches to the subject. Then we shall summarize an argument given by Kijowski where he adduces a set of axioms pertaining to the form that any arrival time distribution must take. From these axioms he obtains a distribution which also frequently recurs in later work. We shall then briefly discuss the backflow effect which, although not the primary topic of this review, has been a topic of increasing interest in studies of arrival time in recent years, and which will occasionally be mentioned in the following chapters.

2.1 Probability current density

In this section we will review the classical approach to arrival times for an ensemble of free particles [9, 10]. The fact that the particles are free means that they will only once cross $x = x_a$, the point at which arrival is to be measured. This simple idea of arrival time, in the absence of complicating potentials, is the focus of this work. In the classical case, there is no problem whatsoever in defining the arrival time distribution. For particles travelling only left-to-right, such that momentum $p > 0$, the time of passage across $x = x_a$ is obviously

$$T = \frac{(x_a - x_0)}{p}. \tag{2.1}$$

If the phase space distribution function of our ensemble is $\rho(x, p, T)$, then the arrival time distribution at $x = x_a$ is given by the classical probability density current

$$J_{cl}(x_a, T) = \int_{-\infty}^{\infty} dx \int_0^{\infty} dp\, \rho(x, p, T) \frac{p}{m} \delta(x - x_a). \tag{2.2}$$

We may consider this current density to be the average over ρ of the following quantity:

$$\mathcal{J}(x, p, x_a) = \frac{p}{m}\delta(x - x_a). \tag{2.3}$$

If x_0 and p_0 are the initial position and momentum respectively, then we may use the classical equation of motion, $x(T) = x_0 + p_0T/m$, and the Liouville theorem, which states that the phase space distribution function remains constant - i.e. $\rho(x(T), p(T), T) = \rho(x_0, p_0, 0)$ - to show show that the average arrival time at $x = x_a$ is given by

$$\langle t_a \rangle = \int_{-\infty}^{\infty} dx_0 \int_0^{\infty} dp_0 \; \rho(x_0, p_0, 0)\frac{m(x_a - x_0)}{p_0}. \tag{2.4}$$

We can see that there is a singularity for p_0. When we consider arrival time operators in chapter 3, we will see that just such a singularity for the $p = 0$ case poses problems for the definition of a self-adjoint operator.

In order to obtain an arrival time distribution in QM, it might now be considered to be reasonable to use the quantum analogue of Eq. 2.2, namely the quantum probability current density for a wave function ψ at $x = x_a$. If we take the position at which arrival time is to be measured as $x_a = 0$, the relevant quantity is given by

$$\Pi(T) = J(0, T) = \frac{1}{2mi}(\psi^*(0, T)\frac{\partial}{\partial x}\psi(0, T) - \psi(0, T)\frac{\partial}{\partial x}\psi^*(0, T)). \tag{2.5}$$

We may now rewrite this quantity as the expectation value of an operator $\hat{\Pi}_J$, such that

$$\begin{aligned} \Pi(T) &= \langle\psi(T)|\hat{\Pi}_J|\psi(T)\rangle \\ &= \langle\psi(T)|\frac{1}{2m}(\hat{p}\delta(\hat{x}) + \delta(\hat{x})\hat{p})|\psi(T)\rangle\,, \end{aligned} \tag{2.6}$$

where $\delta(\hat{x}) = |0\rangle\langle 0|$, $|0\rangle$ in this case being the eigenstate of $\hat{x}$ within eigenvalue 0. $\hat{x}$ and $\hat{p}$ are just the standard position and momentum operators respectively. It is quite clear that Eq. 2.5 can be obtained from Eq. 2.6. It is also manifest that the operator $\hat{\Pi}_J$ is a symmetrized quantum version of the classical quantity $\mathcal{J}(x, p, x_a)$

in Eq. 2.3. The quantum expression will reappear in the following chapters.

2.2 Kijowski's arrival time distribution

Kijowski [8] sought a different means of obtaining an expression for the arrival time distribution, and this was partly motivated by his observation that there is a problem with taking the arrival time distribution $\Pi(T)$ at $x = 0$ to be given by the quantum probability flux. He noted that it is possible for $J(0, T)$ to have negative values, implying that the probability can flow in a direction opposite to the movement of the particle. It is clearly unacceptable that a probability distribution should have negative values. Kijowski, therefore, takes a different approach to obtaining $\Pi(T)$ [8, 10]. He begins by noting that the classical expression $J_{cl}(0, T) = \Pi_{cl}(T)$ in Eq. 2.2 may be obtained by imposing a set of rather intuitive conditions on the family of all possible distributions. Taking $\Pi(\rho)$ to be a function defined over the phase space $\rho(x, p, t)$ as in Eq. 2.2, the set of axioms is:

1. $\Pi_{cl} \geq 0$.

2. $\int_{-\infty}^{\infty} dT \; \Pi_{cl}(T) = 1$.

3. By imposing time reversal, such that $x \to x$ and $p \to -p$, and reflection, such that $x \to -x$ and $p \to -p$, we obtain $\rho_1(x, p, t) = \rho(-x, p, t)$. We must take $\Pi_{cl}(\rho_1) = \Pi_{cl}(\rho)$, which fixes $x_a = 0$ as the arrival point in this case.

4. The correct arrival time distribution is obtained by taking the distribution with the minimum value of the variance $\int_{-\infty}^{\infty} dT \; (T - \langle T \rangle)^2 \Pi(\rho)$.

$\langle T \rangle$ is the average arrival time. We may think of condition 4 as a stipulation that we take the correct distribution as that which corresponds to an ideal detector measuring the arrival time as accurately as possible. This is proven in the final section of [8].

The main idea leading to Kijowski's derivation of an arrival time distribution is now as follows: instead of simply quantizing the probability current density, look for a unique distribution in the quantum case which is derived by quantum versions of the above axioms. Accordingly, Kijowski defines a set of conditions on the allowed distributions which are quantized versions of those above. As in Eq. 2.6 above, possible distributions are of the form $\Pi[\psi(T)] = \langle\psi(T)|\hat{\Pi}|\psi(T)\rangle$, where the $\psi(T)$ are wave functions with only positive momentum components. We then have the axioms:

1. $\Pi[\psi(T)] \geq 0$.

2. $\int_{-\infty}^{\infty} dT\ \Pi[\psi(T)] = 0$.

3. Taking $\hat{\pi}$ to be the parity operator, and $\hat{\mathcal{R}}$ to be the time reversal operator, then we require that $\langle\psi(0)|\hat{\Pi}|\psi(0)\rangle = \langle\hat{\pi}\hat{\mathcal{R}}\psi(0)|\hat{\Pi}|\hat{\pi}\hat{\mathcal{R}}\psi(0)\rangle$.

4. The correct arrival time distribution is obtained by taking the distribution with the minimum value of the variance $\int_{-\infty}^{\infty} dT\ (T - \langle T\rangle)^2 \Pi(\psi(T))$.

Again, Kijowski gives a mathematical proof showing that the appropriate operator $\hat{\Pi}_K$ satisfies

$$\begin{aligned} \Pi_K(T) &= \langle\psi(T)|\hat{\Pi}_K|\psi(T)\rangle \\ &= \langle\psi(T)|\frac{1}{m}(\hat{p}^{\frac{1}{2}}\delta(\hat{x})\hat{p}^{\frac{1}{2}})|\psi(T)\rangle\,. \end{aligned} \tag{2.7}$$

Similar arguments may be made for the case of states of purely negative momentum, and both may be added together in order to give a total distribution. Explicitly this may be written

$$\Pi_K(T) = \frac{1}{2\pi m}\left|\int_0^\infty dp\sqrt{p}e^{-iTp^2/(2m)}\psi(p)\right|^2 + \frac{1}{2\pi m}\left|\int_{-\infty}^0 dp\sqrt{-p}e^{-iTp^2/(2m)}\psi(p)\right|^2 \tag{2.8}$$

This distribution is cited widely in the literature on arrival time. Often, attempts to define arrival time will arrive at the same result for $\Pi(T)$, and indeed the similarity to $\Pi_K(T)$ is often used as a benchmark to judge the success or otherwise of the method by which the distribution was derived. We shall see numerous examples of this in the coming chapters.

As an aside, one must not mistake the operator $\Pi_K(T)$, nor the operator defined above as the quantization of the probability density current, for a time operator $\hat{T}$. If we examine the average arrival time $\langle T \rangle$, we find that if there exists a time operator, then [9]

$$\langle T \rangle = \langle \psi(0)|\hat{T}|\psi(0)\rangle = \int_{-\infty}^{\infty} dT\, \hat{\Pi}(T)T \tag{2.9}$$

so that they are clearly not the same thing.

In summary then, we have two flux operators whose expectation values for $|\psi(T)\rangle$ give possible arrival time distributions:

$$\hat{\Pi}_J = \frac{1}{2m}(\hat{p}\delta(\hat{x}) + \delta(\hat{x})\hat{p}) \tag{2.10}$$

and

$$\hat{\Pi}_K = \frac{1}{m}(\hat{p}^{\frac{1}{2}}\delta(\hat{x})\hat{p}^{\frac{1}{2}}). \tag{2.11}$$

We can think of these as two distinct symmetrizations of the classical expression in Eq. 2.3 [10].

We now notice a feature of this distribution $\Pi_K(T)$. Recalling the expression for the time evolution of a state, $|\psi(T)\rangle = e^{-i\hat{H}T}|\psi(0)\rangle$, and taking $\hat{H}$ to be the free particle Hamiltonian $\hat{p}^2/(2m)$, we notice that

$$\begin{aligned}\Pi_K(T, \psi(t)) &= \langle \psi(0)| e^{i\hat{H}t}\hat{\Pi}_K e^{-i\hat{H}t}|\,|\psi(0)\rangle\rangle \\ &= \Pi_K(T + t, \psi(0)).\end{aligned} \tag{2.12}$$

This expresses the so-called *covariance* of the arrival time distribution, which means that the probability for a state $|\psi(0)\rangle$ to arrive at time $T + t$ is the same as the probability for a time-evolved state $|\psi(t)\rangle$ to arrive at the earlier time T [11]. This is a very intuitive condition which any arrival time distribution must meet be in order to be a physically interesting distribution. We will see in the following chapters that whether or not a distribution satisfies this covariance condition is important for determining the validity of that distribution.

2.3 The backflow effect

As we noted there in section 2.2, it is possible for the probability current density to have negative values, which is known as the backflow effect. This means that for a particle with entirely positive momenta located in $x < 0$, there may be times at which the probability of detection in $x < 0$ increases rather than decreases. The possible negativity of $J(0,T)$ is the reason that Allcock rejected it as a candidate for the arrival time distribution, since it seems unreasonable to suggest that a *probability* for arrival within a certain time range could be negative [6, 7]. However, in the following chapters we will find that various approaches to obtaining an arrival time distribution arrive at the probability density current. As a result, this distribution will be taken as an approximation to the arrival time distribution or, in the decoherent histories approach, a time uncertainty will be introduced to ensure that this quantity remains positive. It is therefore useful to better understand how this quantity can be negative, and to understand the magnitude of the effect.

Firstly, let us define the quantum probability flux, which measures how much probability flows across $x = 0$ in a given time interval. We can define this to be

$$F(t_1, t_2) = \int_{t_1}^{t_2} dT \; J(0,T). \tag{2.13}$$

In analogy with Eq. 2.2, which gives $J_{cl}(0,T)$ for the classical case in terms of the classical phase space distribution function $\rho(x,p,T)$, we can express the quantum flux in terms of the Wigner quantum phase space distribution function as follows:

$$F(t_1, t_2) = \int_{t_1}^{t_2} dT \int dp \; dx \; W(x,p,T) \frac{p}{m} \delta(x). \tag{2.14}$$

It is well-known that the Wigner function is not necessarily positive. We can clearly see, then, that the flux need not be positive either.

To quantify this effect, it is customary to treat it as eigenvalue problem. Bracken and Melloy [12] did this by taking the standard form for time-dependent wave functions:

$$\psi(0,T) = \frac{1}{2\pi}\int_0^\infty dp\ \phi(p)e^{-ip^2T/(2m)}. \tag{2.15}$$

Using this with the formula for $J(0,T)$ in Eq. 2.5 gives

$$J(0,T) = \frac{1}{4\pi m}\int_0^\infty dp \int_0^\infty dq\ (p+q)e^{i(p^2-q^2)T/(2m)}\phi^*(p)\phi(q). \tag{2.16}$$

Using Eq. 2.13, we may then write

$$F(0,T) = \int_0^\infty dp \int_0^\infty dq\ \phi^*(p)K(p,q)\phi(q), \tag{2.17}$$

where

$$K(p,q) = \frac{i}{2m}\left(\frac{1-e^{i(p^2-q^2)T/(2m)}}{p-q}\right). \tag{2.18}$$

Now Bracken and Melloy maximize $F(0,T)$ subject to normalization constraints, by using Lagrangian multipliers, and obtain the equation

$$\int_0^\infty dq\ K(p,q)\phi(q) = \lambda\phi(p). \tag{2.19}$$

Hence we have an eigenvalue equation, and it can be shown from Eq. 2.17 that $\lambda = F(0,T)$. Letting $\Phi(p) = e^{-ip^2T/(4m)}\phi(p)$ gives us

$$\frac{1}{\pi}\int_0^\infty dq\ \frac{\sin((p^2-q^2)T/4m)}{p-q}\Phi(q) = \lambda\Phi(p). \tag{2.20}$$

This is difficult to solve analytically, but may be solved numerically to give $-0.038452 \leq \lambda \leq 1$ [13]. Hence we have calculated a small but non-zero backflow effect.

2.4 The Zeno effect

Finally, we describe briefly the *Zeno effect*, which will be important for discussions in chapters 4 and 5 [14]. The Zeno effect is a peculiar effect in quantum mechanics whereby a system which is rapidly monitored is more likely to remain in its initial state. To understand this mathematically, we shall use the density matrix formalism.

Consider an initial state ρ which evolves unitarily over a period of time t. Now recall that a measurement on a system corresponds to applying a projection operator to a state, so that a measurement of ρ corresponds to $\rho \rightarrow P\rho P$. Assume that a state is initially in a Hilbert subspace $\mathcal{H}_s$ of $\mathcal{H}$, and take P to be the projector onto the subspace. Whether or not the system remains in the subspace is now measured at regular intervals ϵ, where $N\epsilon = t$, giving us

$$\rho \rightarrow Pe^{-i\hat{H}\epsilon} \ldots Pe^{-i\hat{H}\epsilon} P\rho P e^{i\hat{H}\epsilon} P \ldots e^{i\hat{H}\epsilon} P \tag{2.21}$$

so that the probability for the system to still be found in $\mathcal{H}_s$ at the end of the period is

$$p(t) = Tr(Pe^{-i\hat{H}\epsilon} \ldots Pe^{-i\hat{H}\epsilon} P\rho P e^{i\hat{H}\epsilon} P \ldots e^{i\hat{H}\epsilon} P). \tag{2.22}$$

Defining $\overline{P} = 1 - P$, we obtain for small ϵ

$$Pe^{-i\hat{H}\epsilon} P \approx Pe^{-iP\hat{H}P\epsilon} \left(1 - \frac{\epsilon^2}{2} P\hat{H}\overline{P}\hat{H}P\right) \tag{2.23}$$

so that in this limit we have

$$(Pe^{-i\hat{H}\epsilon} P)^N = Pe^{-iP\hat{H}Pt} \left(1 - \frac{\epsilon^2}{2} P\hat{H}\overline{P}\hat{H}P\right)^N, \tag{2.24}$$

Substituting this into Eq. 2.22, we have to leading order:

$$p(t) = 1 - \frac{N\epsilon^2}{t_z^2}, \tag{2.25}$$

where t_z is the Zeno time given by $(\left\langle \hat{H}\overline{P}\hat{H} \right\rangle)^{-1/2}$. The important point here is that as $\epsilon \rightarrow 0$, which corresponds to continuous observation of the system, the probability for remaining in this Hilbert subspace goes to one. We can see that this might be a significant effect for measurements of arrival time, where the Hilbert subspace corresponds to a particle remaining on one side of the arrival point, and indeed we will find that it is important when we consider decoherent histories in chapter 5.

3 Time Operators and eigenstates

3.1 Introduction

In this chapter we will look at the issue of arrival time operators and their eigenvalues. We will begin by explaining in more detail Pauli's assertion that it is impossible to define a self-adjoint time operator canonically conjugate to the Hamiltonian, given the semi-bounded spectrum of the Hamiltonian. We will then explain the difference between self-adjoint and symmetric operators, and we will examine the spectrum of the symmetric Aharonov-Bohm arrival time operator, observing that the eigenvalues are not orthogonal. Next we will look in detail at Allcock's argument that even in the presence of a source, which enables us to introduce effective negative energy states, it is impossible for there to be orthogonal arrival time eigenstates. We will then examine various attempts which have nonetheless been made to obtain an arrival time operator from which a distribution may be derived. Next, we will see a refutation of Pauli's argument, and an attempt to define a self-adjoint operator for a particle confined particle. Finally, we discuss why these attempts may be inadequate due to their excessively ideal nature.

3.1.1 Pauli's theorem

If one wants to deal with the time of arrival mathematically in quantum mechanics, a natural first approach is to attempt to define a self-adjoint time of arrival operator $\hat{T}$. Just as with position and momentum operators, we should then be able to easily define a commutation relation between time and energy, such as

$$[\hat{H}, \hat{T}] = i, \tag{3.1}$$

and hence to define an uncertainty relation such as that which pertains between position and momentum. Furthermore, due to the spectral theorem, which applies for self-adjoint operators, we should be able to calculate orthogonal eigenstates of the operator $|T\rangle$, and hence obtain a time of arrival distribution for a state $|\psi\rangle$ via

$$\Pi(T) = |\langle T|\psi\rangle|^2 = \langle\psi|\,\hat{P}(T)\,|\psi\rangle\,, \tag{3.2}$$

where $\hat{P}(T)$ is the projector $|T\rangle\langle T|$ onto the subspace of the Hilbert space corresponding to detection at time T. We will now examine why this method doesn't work for arrival time operators, necessitating the more complicated proposals presented below.

Pauli was one of the earliest people to note the problematic nature of time as an observable in quantum mechanics. In the 1930s he noted that, due to the bounded nature of the spectrum for the Hamiltonian in the absence of a potential, where all the eigenvalues must be positive, it is not possible to define a time operator which will satisfy the commutation relation in Eq. 3.1. Delgado and Muga showed this [3] by noting that using Eq. 3.1 one can show by induction that

$$[\hat{H}, \hat{T}^n] = in\hat{T}^{n-1}, \tag{3.3}$$

for $n \geq 0$. Hence, if we have a parameter ϵ with dimensions of energy, we can show that

$$[\hat{H}, e^{i\epsilon\hat{T}}] = -\epsilon e^{i\epsilon\hat{T}} \tag{3.4}$$

and hence for eigenstates of the hamiltonian $|E\rangle$ we have

$$\hat{H}e^{i\epsilon\hat{T}}\,|E\rangle = (E-\epsilon)e^{i\epsilon\hat{T}}\,|E\rangle\,. \tag{3.5}$$

We can also reverse the argument to show Eq. 3.1 from Eq. 3.5, and hence show that the existence of the commutation relation between the Hamiltonian and the self-adjoint time operators is equivalent to the statement that the time operator generates unitary energy translations. But this implies that we could apply this to produce energy eigenstates $|E-\epsilon\rangle$ corresponding to arbitrary energy values ranging across the entire real line, which contradicts the required semi-boundedness of the energy spectrum. Hence one cannot construct a self-adjoint time operator satisfying Eq. 3.1.

This was perceived to be an insurmountable problem for putting time on the same footing as other observables in the mathematical formalism. By the spectral theorem, if the time operator is not self-adjoint, then its eigenvalues are not orthogonal, a fact which we explicitly demonstrate for a particular time operator below. In the past, such non-orthogonality of eigenstates was seen as being inconsistent with quantum observables. Allcock noted, for example that when measuring some observable, two different wave functions corresponding to different measurement eigenstates must be distinguishable from another and hence must not overlap in the final state measured by an apparatus. This implies they must be orthogonal [5]. But since evolution according to the Schrödinger equation conserves inner products, this implies that the initial eigenstates were also orthogonal. Hence an operator which is not self-adjoint cannot describe physically interesting and distinguishable states, and is hence useless. Despite these issues, we will see how attempts have been made to nonetheless define a useful arrival time operator. But first we will briefly discuss some important subtleties relating to self-adjoint operators.

3.1.2 Symmetric and self-adjoint operators

A fact often glossed over in elementary introductions to QM is that a symmetric operator is not the same thing as a self-adjoint operator. We first define the domain of an operator, $D(\hat{A})$, which is the subset of eigenfunctions or eigenstates upon which the operator $\hat{A}$ can act. A symmetric operator is one such that $\langle\phi|\hat{A}\psi\rangle = \langle\hat{A}\phi|\psi\rangle$. The adjoint of an operator A is given by $A^\dagger$ where $\langle\phi|\hat{A}^\dagger\psi\rangle = \langle\hat{A}\phi|\psi\rangle$. If the domains $D(\hat{A})$ and $D(\hat{A}^\dagger)$ are not the same, it is possible for an operator to be symmetric without being self-adjoint. An explicit example is provided in [15]: the domain of the operator $\hat{p}$, when conjugate to $\hat{x}$, defined only for positive values of x, is not the same as the domain of its adjoint. In fact, the domain of $\hat{p}^\dagger$ in this case is bigger than that of $\hat{p}$, since $\hat{p}^\dagger$ also has non-degenerate eigenstates with eigenvalues containing a positive imaginary part . Only if it is specifically demonstrated that $\hat{A} = \hat{A}^\dagger$, which is to say that the domains are equal, are we assured that the operator is self-adjoint and that the eigenvalues will be orthogonal [4]. Self-adjoint operators will have real, complete, and orthogonal eigenvalues, which may not be the case for symmetric operators. For non-self-adjoint operators such as $\hat{p}$ above, the *deficiency indices* are two numbers which give the dimension of the space of eigenstates with positive and negative imaginary parts respectively. This operator consequently has deficiency indices $(1, 0)$. According to the theory of von Neumann, only in the case that the deficiency indices are equal can one extend the domain of the operator in such a way that it becomes self-adjoint. Such self-adjoint extensions do not exist for $\hat{p}$. In the case that self-adjoint extensions do exist, we may consider an operator to be effectively self-adjoint. In the case that such extensions do not exist, we refer to a symmetric operator as *maximally symmetric* [15].

3.1.3 Aharonov-Bohm operator

Aharonov and Bohm were among the first to look at a candidate for the time operator [16], which has been (with the addition of a minus sign) studied as a possible arrival time operator [9, 11, 15]. The form of this operator is

$$\hat{T} = -\frac{m}{2}\left(\hat{x}\frac{1}{\hat{p}} + \frac{1}{\hat{p}}\hat{x}\right), \tag{3.6}$$

where $\hat{x}$ and $\hat{p}$ are the standard operators for position and momentum respectively. This was obtained by simply symmetrizing the classical expression for time of arrival - the symmetrization being necessary in order to ensure the operator is Hermitian. We can see that, if we take a free particle Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m}$, this arrival time operator satisfies the commutation relation in Eq. 3.1. However, as we shall see, the eigenstates of this operator are non-orthogonal, veryifying that this time operator is not self-adjoint. Egusquiza and Muga solve for the eigenvalues of (3.6) by going to the momentum representation, where $\hat{p} \to p$ and $\hat{x} \to i\frac{\partial}{\partial p}$. This is the natural choice since it makes it easier to deal with $\hat{p}^{-1}$. The result is that

$$\hat{T} \to \frac{im}{2}\left(\frac{1}{p^2} - \frac{2}{p}\frac{\partial}{\partial p}\right). \tag{3.7}$$

There is clearly a singularity at $p = 0$. This suggests that there are constraints on the functions in the Hilbert space upon which the operator can act. To understand exactly what the domain of this operator is, we must impose some conditions. Firstly, the operator must obey the property that when it operates on a state $|\psi\rangle$, $\hat{T}\,|\psi\rangle$ remains in the same Hilbert space, and is therefore square integrable across the entire spectrum of p. Hence, we get two possible conditions: either $\psi(p)/p^{3/2} \to 0$ as $p \to 0$, or $\psi(p) \sim p^{1/2}$, for functions $\psi(p)$ in the operator's domain. Secondly, if we impose the condition that $\hat{T}$ is symmetric, such that $\langle\psi|\hat{T}\psi\rangle = \langle\hat{T}\psi|\psi\rangle$ for any $|\psi\rangle$ in the domain of $\hat{T}$, then it is clearly the case that $\psi(p) \sim p^{1/2}$ is untrue, and we are left with the condition that $\psi(p)/p^{3/2} \to 0$ as $p \to 0$.

The eigenvalues of the operator $\hat{T}$ can now be shown to be [9, 11, 15, 17]

$$\langle p|T,\alpha\rangle = \psi_\alpha{}^{(T)}(p) = \left(\frac{\alpha p}{2\pi m}\right)^{\frac{1}{2}} e^{ip^2T/(2m)}\theta(\alpha p) \tag{3.8}$$

where $\alpha = \pm$ and $p \neq 0$. We can confirm that these are the eigenstates by applying the operator $\hat{T}$ to them:

$$\begin{aligned}
\hat{T}\psi_\alpha{}^{(T)}(p) &= \frac{im}{2p^2}\psi_\alpha{}^{(T)}(p) - \frac{im}{p}\frac{\partial}{\partial p}\psi_\alpha{}^{(T)}(p) \\
&= \frac{im}{2p^2}\psi_\alpha{}^{(T)}(p) - \frac{im}{p}\left[\frac{1}{2}\left(\frac{\alpha}{2\pi m p}\right)^{\frac{1}{2}} e^{ip^2T/(2m)}\theta(\alpha p)\right. \\
&\qquad \left. + \left(\frac{\alpha p}{2\pi m}\right)^{\frac{1}{2}} \frac{ipT}{m} e^{ip^2T/(2m)}\theta(\alpha p) + \left(\frac{\alpha p}{2\pi m}\right)^{\frac{1}{2}} e^{ip^2T/(2m)}\alpha\delta(\alpha p)\right] \\
&= -\frac{im}{p}\left(\frac{ipT}{m}\left(\frac{\alpha p}{2\pi m}\right)^{\frac{1}{2}} e^{ip^2T/(2m)}\theta(\alpha p)\right) \\
&= T\psi_\alpha{}^{(T)}(p).
\end{aligned}$$

This operator is also consistent with the covariance condition described in section

2, since

$$e^{-i\hat{H}t}\,|T,\alpha\rangle = |T-t,\alpha\rangle\,. \tag{3.9}$$

For an operator to be self-adjoint, it must not only have real eigenvalues, its eigenstates must also be complete and orthogonal. It can be shown that,

$$\sum_{\alpha}\int_{-\infty}^{\infty} dT\overline{\psi_{\alpha}}^{(T)}(p)\psi_{\alpha}{}^{(T)}(p') = \delta(p-p') \tag{3.10}$$

and so these eigenvalues provide a resolution of the identity and are therefore complete. The overlap of two eigenstates is given in [17], [11] and [15]. It is given by

$$\begin{aligned}\langle T',\alpha'|T,\alpha\rangle &= \frac{\delta_{\alpha,\alpha'}}{2\pi m}\int_{0}^{\infty} dk^{2}e^{ik^{2}(T-T')/(2m)} \\ &= \frac{1}{2}\delta_{\alpha,\alpha'}\left(\delta(T-T') + \frac{i}{\pi}P\left(\frac{1}{T-T'}\right)\right),\end{aligned} \tag{3.11}$$

where $P()$ refers to a quantity called the *principal part* which comes from a standard result for this type of integral in complex analysis. The important point is that the RHS is not simply $\delta_{\alpha,\alpha'}\delta(T-T')$, and that the eigenstates are therefore not orthogonal. Hence, we have verified that the operator is not self-adjoint. Due to the absence of orthogonal eigenstates, we cannot define projectors as in Eq. 3.2, and hence dealing with distributions and probabilities in the standard quantum mechanical fashion is not possible. Furthermore, as demonstrated in [15], the deficiency indices of this operator are $(2,0)$, so that no self-adjoint extensions exist. It is therefore a maximally symmetric operator. We will see how this arrival time operator, and variants of it, have nonetheless been the starting point for many attempts to define a suitable arrival time operator ever since. In many cases we will see that the result of these efforts is to obtain Kijowski's distribution, but this time in the context of the traditional mathematical formalism of QM.

3.2 Allcock's consideration of the problem with sources

Allcock was aware of the non-orthogonality of the Aharonov-Bohm operator, and suggested that this non-orthogonality applied to any possible arrival time eigenfunctions due to the semi-bounded nature of the energy spectrum [5]. He concludes that the only way of salvaging the situation is to somehow introduce negative energy components of the spectrum. This is his motivation for introducing a source for the incoming particles at some point x_s, where $x_s < x_0 < 0$, in an attempt to derive an expression for the arrival time at $x = 0$ of particles travelling left-to-right. This source term $\rho(x,t)$ leads to the Schrödinger equation.

$$\frac{i\partial\psi}{\partial t} + \frac{1}{2}\frac{\partial^2\psi}{\partial^2 x} = \rho(x,t) \tag{3.12}$$

such that $\rho(x,t)$ is nonzero in the region of the source, but zero outside of it - e.g. in the region $x > x_0$.

By introducing a source term, he is also introducing an additional time boundary condition, namely the condition since the particle is liberated from the source at a particular time, there must be a time t_0 before which the wave function satisfies $\psi = 0$ in the region $x > x_0$. This boundary condition leads to the following solution for the Schrödinger equation in the region $x > x_0$:

$$\psi(x,t)_{x\geq x_0} = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dE \, \frac{\psi(E)}{(2E)^{1/4}} e^{(i(2E)^{1/2}x - iEt)}, \tag{3.13}$$

where $\psi(E)$ is in the energy representation. Allcock assumes that the source term $\rho(x,t)$ can be modified in order to match any arbitrarily chosen $\psi(E)$, such that his analysis is quite general. An important point about Eq. 3.13 is that there are now negative energy values in the integral, implying that the total wave function includes contributions from exponentially decaying evanescent waves, as can be seen by entering negative E values in the factor $e^{(i(2E)^{1/2}x)}$ in the intergral above.

Allcock now shows that in the limit as $t \to \infty$, for any finite $a > x_0$, we have that

$$\lim_{t\to+\infty} \int_{x_s}^{a} dx\ |\psi(x,t)|^2 = 0, \tag{3.14}$$

and

$$\lim_{t\to+\infty} \int_{a}^{\infty} dx\ |\psi(x,t)|^2 = \int_{0}^{\infty} dE\ |\psi(E)|^2. \tag{3.15}$$

Hence in the limit as $t \to \infty$, the probability that that the particle has arrived at any point a finite distance from the source, such as the point at which arrival time is measured, is

$$P(\infty) = \int_{0}^{\infty} dE\ |\psi(E)|^2. \tag{3.16}$$

Obviously Eq. 3.14 is an expression of the intuitive fact that a wave function's tail goes to zero in the limit as $x \to \infty$, so that as the centre of the wave function goes towards ∞, the norm left in any finite region goes to zero. Note that the 0 in the lower limit of the integral in Eq. 3.16 is crucial. Now, we come to the matter of arrival time eigenstates. If we now posit the existence of orthogonal arrival time eigenfunctions $\psi^{(T)}(E) = \langle E|T\rangle$, we may write our energy representation wave functions as

$$\psi(E) = \langle E|\psi\rangle == \int_{-\infty}^{\infty} dT\ c(T)\psi^{(T)}(E). \tag{3.17}$$

where $c(T)$ is some set of c-numbers. Now calculating the total norm of a state $|\psi\rangle$, which may be of course interpreted as the total time integral of the arrival time distribution $\Pi(T)$ for particles with $p > 0$ originating at $x < 0$, we obtain

$$\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} dE\ |\psi(E)|^2 = \int_{-\infty}^{\infty} dT\ |c(T)|^2 = \int_{-\infty}^{\infty} dT\ \Pi(T). \tag{3.18}$$

This quantity can be identified with $P(\infty)$, and is clearly positive for all values. In this case we note that due to the introduction of negative energy terms, the integral ranges over the entire real line. However, for $P(\infty)$ as defined in Eq. 3.16, we may obtain $P(\infty) = 0$ for waves with $E < 0$. Allcock argues that this contradiction implies the impossibility of defining orthogonal arrival time eigenstates even for the case where we take the particles as originating at a source.

Allcock uses this argument to suggest that an operational approach must be considered in order to define an arrival time distribution theoretically. It is worth pointing out, however, that Muga and Leavens disagree with this argument since they suggest that the evanescent waves associated with the $E < 0$ contribution, although tied very closely to the source location, may in fact contribute to the arrival time distribution through tunnelling in the infinite time limit [9]. As we see in the following sections, moreover, other authors not only accept the possibility of orthogonal arrival time eigenstates, but in fact explicitly calculate them.

3.3 Attempts to define self-adjoint operators

3.3.1 Grot, Rovelli and Tate

In recent years, one of most discussed attempts to construct a self-adjoint arrival time operator has been that of Grot, Rovelli and Tate [18]. These authors do not use exactly Aharonov-Bohm operator defined above, but rather a different symmetrization of the classical expression, namely

$$\hat{T} = -m\frac{1}{\sqrt{\hat{p}}}\hat{x}\frac{1}{\sqrt{\hat{p}}} \rightarrow -im\frac{1}{\sqrt{p}}\frac{\partial}{\partial p}\frac{1}{\sqrt{p}}. \tag{3.19}$$

It turns out, however, that this operator gives the same eigenstates as in Eq. 3.8, and hence that the two operator expressions are equivalent [9, 18]. Grot et al. trace the non-orthogonality of the expression above to the singularity when $p = 0$, and they therefore modify the operator so that this singularity no longer exists. While this may no longer represent the standard quantity for time of arrival in the classical limit, it can be defined so as to be arbitrarily close to it, which the authors consider to be satisfactory. Their suggested definition for the arrival time operator is

$$\hat{T}_\epsilon = -im\sqrt{f_\epsilon(p)}\frac{\partial}{\partial p}\sqrt{f_\epsilon(p)}, \tag{3.20}$$

where

$$f_\epsilon(p) = \frac{1}{p}, \qquad |p| > \epsilon;$$
$$f_\epsilon(p) = \frac{p}{\epsilon^2}, \qquad |p| < \epsilon. \tag{3.21}$$

We can see that there is continuity of the function at $p = \epsilon$, but that the singularity at $p = 0$ is now removed. We may now find the eigenstates which are given by

$$\langle p|T,\alpha\rangle_\epsilon = \psi_\alpha{}^{(T)}(p) = \left(\frac{1}{2\pi m}\right)^{\frac{1}{2}} \frac{1}{f_\epsilon(p)} \exp\left(\frac{iT}{m}\int_{\alpha\epsilon}^{p} dp' \frac{1}{f_\epsilon(p')}\right)\theta(\alpha p). \tag{3.22}$$

ϵ may be arbitrarily small, and when doing calculations we may later restrict ourselves to the values of $|p| > \epsilon$ (labelled with the subscript o). Inserting the value of $f_\epsilon(p)$ from Eq. 3.21 and carrying out the integral, we obtain

$$\psi_{\alpha,o}{}^{(T)}(p) = \left(\frac{\alpha p}{2\pi m}\right)^{\frac{1}{2}} \exp\left(\frac{iT}{2m}(p^2 - \epsilon^2)\right)\theta(\alpha p). \tag{3.23}$$

However, we also note that for $|p| < \epsilon$ (labelled with the subscript ϵ we have

$$\psi_{\alpha,\epsilon}{}^{(T)}(p) = \left(\frac{\alpha p}{2\pi m}\right)^{\frac{1}{2}} \exp\left(\frac{iT}{m}(\epsilon^2 \ln|p|/\epsilon)\right)\theta(\alpha p). \tag{3.24}$$

To show that the operator is self-adjoint, we must demonstrate that the eigenstates in Eq. 3.22 are complete and orthogonal. To do this, we must introduce a new coordinate system which ranges from $-\infty$ to $+\infty$:

$$z^\alpha(p) = \int_{\alpha\epsilon}^{p} dp' \frac{1}{f_\epsilon(p')}. \tag{3.25}$$

We may then rewrite the eigenstates as

$$\psi_{\alpha,\epsilon}{}^{(T)}(p) = \left(\frac{1}{2\pi m}\right)^{\frac{1}{2}} \frac{1}{f_\epsilon(p(z^\alpha))} \exp\left(\frac{iTz^\alpha}{m}\right)\theta(\alpha p), \tag{3.26}$$

so that the overlap of two states is now

$$\langle T',\alpha'|T,\alpha\rangle = \delta_{\alpha,\alpha'} \int_{-\infty}^{\infty} dz^\alpha f_\epsilon(p(z^\alpha))\overline{\psi_\alpha}^{(T')}(p)\psi_\alpha{}^{(T)}(p). \tag{3.27}$$

Due to the presence of the function f_ϵ, which comes in due to the coordinate change defined above, we now have the simple form

$$\langle T', \alpha' | T, \alpha \rangle = \delta_{\alpha,\alpha'} \int_{-\infty}^{\infty} dz^\alpha e^{i(T-T')z^\alpha/m} = \delta_{\alpha,\alpha'} \delta(T - T'), \tag{3.28}$$

which demonstrates that the eigenvalues in this case are orthogonal. The absence of more complicated terms is mathematically due to the fact that z ranges over the entire real line, and not just the half line as in Eq. 3.11.

We may also show that these eigenstates are complete by noting that

$$\begin{aligned} \sum_\alpha \int_{-\infty}^{\infty} dT \overline{\psi_\alpha}^{(T)}(p) \psi_\alpha{}^{(T)}(p') &= \sum_\alpha \frac{\delta(z^\alpha(p) - z^\alpha(p'))}{\sqrt{f_\epsilon(p)}\sqrt{f_\epsilon(p')}} \\ &= \sum_\alpha \frac{\delta(p - p')\theta(\alpha p)}{|\partial z^\alpha / \partial p| \sqrt{f_\epsilon(p)} \sqrt{f_\epsilon(p')}} \\ &= \delta(p - p'), \end{aligned} \tag{3.29}$$

where in the second line we used the fact that $\delta(f(x)) = \delta(x - x_0)/|f'(x_0)|$. Hence the modified operator presented by Grot et al. has real eigenvalues, and its eigenstates are complete and orthogonal. Hence it is a self-adjoint operator.

From these eigenstates we can now obtain the arrival time distribution via

$$\Pi(T) = \sum_\alpha |\langle T, \alpha | \psi \rangle|^2, \tag{3.30}$$

which is of course summed over α due to the double degeneracy of the eigenvalues at each T, corresponding to particle moving in the positive and negative directions. Grot et al. now argue that since we may make ϵ as small as we wish, we may effectively ignore the states of $|p| < \epsilon$ and only use the states of Eq. 3.23. This is important because under time translation the states $\psi_{\alpha,\epsilon}^{(T)}(p)$ are not consistent with the covariance condition for the time operator. If we limit ourselves to a particle moving in the positive direction, the arrival time distribution for detection at $x = 0$

is

$$\Pi(T) = \frac{1}{2\pi m} \left| \int_0^\infty dp \sqrt{p} e^{-iTp^2/(2m)} \psi(p) \right|^2, \tag{3.31}$$

which is basically Kijowski's distribution. Grot et al. further bolster their argument that this is a plausible arrival time operator by demonstrating that a Gaussian wave packet's arrival time is centred around the classical arrival time value, and by demonstrating that the operator's commutation relation with the Hamiltonian produces a plausible uncertainty relation. On the basis of this evidence, there may be very good grounds for considering this operator a plausible candidate for a definitive arrival time operator.

However, Oppenheim, Reznik and Unruh [19] examined this operator further and reached some unsettling conclusions. In order to examine it, they also looked at a Gaussian wave packet, and examined the norm of the state. Oppenheim et al. noted that the eigenstates are not normalizable, so that it is necessary to introduce states which are superpositions of eigenstates, with a spread of arrival states Δ. These are given by

$$|\tau_\Delta, \alpha\rangle = N \int dT \, |T, \alpha\rangle \, e^{-\frac{(T-\tau)^2}{\Delta^2}}. \tag{3.32}$$

where $N = \frac{1}{(2\pi^3)^{1/4}\sqrt{\Delta}}$. Now, restricting ourselves to states with momentum going to the right, $\alpha = +$, we can find $\langle x|\tau_\Delta, +\rangle$ not only as a function of position, but also as a function of time, by applying the free particle propagator $e^{\frac{-ip^2 t}{2m}}$ where $t = 0$ is the classical time of arrival. The result can be divided between states of $|p| < \epsilon$ and $|p| > \epsilon$, and without loss of generality setting $\tau = 0$ we obtain

$$\begin{aligned}
\tau^+(x,t) = \langle x|\tau_\Delta, +\rangle &= N \int dT \, \langle x| e^{\frac{-ip^2 t}{2m}} |T, +\rangle \, e^{-\frac{T^2}{\Delta^2}} \\
&= N \int_0^\epsilon dT dp e^{-\frac{T^2}{\Delta^2}} e^{\frac{-ip^2 t}{2m}} e^{ipx} \psi_{\alpha,\epsilon}{}^{(T)}(p) + N \int_\epsilon^\infty dT dp e^{-\frac{T^2}{\Delta^2}} e^{\frac{-ip^2 t}{2m}} e^{ipx} \psi_{\alpha,o}{}^{(T)}(p) \\
&= \tau_\epsilon^+(x,t) + \tau_o^+(x,t).
\end{aligned} \tag{3.33}$$

The calculation of the term $\tau_\epsilon^+(x,t)$ is quite complicated, but it is shown in [11] and [19] that as $\epsilon \to 0$, its modulus squared vanishes at $x = 0$ and $t = 0$ - i.e. the probability of detection of these states is zero at the time of arrival. However as we saw above, these states are excluded when using the operator of Grot et al. to determine the arrival time distribution, and hence the fact that these correspond to a probability that the particle is undetected may by irrelevant, if their contribution to the overall norm is small. Let us then calculate the norm of this state at the time of arrival $t = 0$, which may be divided into two sections corresponding to superpositions of states with $|p| < \epsilon$ and $|p| > \epsilon$:

$$\begin{aligned}\int dx |\langle x|\tau_\Delta, +\rangle|^2 &= N^2 \int dx |dTdp e^{-\frac{T^2}{\Delta^2}} e^{ipx} {\psi_{\alpha,\epsilon}}^{(T)}(p)|^2 + N^2 \int dx |dTdp e^{-\frac{T^2}{\Delta^2}} e^{ipx} {\psi_{\alpha,o}}^{(T)}(p)|^2 \\ &\equiv {N_\epsilon}^2 + {N_o}^2.\end{aligned} \tag{3.34}$$

We may now explicitly calculate the values of the norms ${N_\epsilon}^2$ and ${N_o}^2$. Specifically, using the expression for $\langle p|T, \alpha\rangle_\epsilon$ given in Eq. 3.22, a calculation of ${N_\epsilon}^2$ gives us

$${N_\epsilon}^2 = \frac{N^2}{2\pi m} \int_0^\epsilon dpdp' \int dTdT'dx \sqrt{p}\sqrt{p'} e^{\frac{(-T^2 - T'^2)}{\Delta^2}} e^{\frac{i\epsilon^2}{m}\left(T \ln \frac{p}{\epsilon} - T' \ln \frac{p'}{\epsilon}\right)} \tag{3.35}$$

Carrying out the integral over x, then p', completing the square in the exponential, doing the integral over T and T', and changing variables gives us

$${N_\epsilon}^2 = \frac{N^2 \epsilon^2 \Delta^2 \pi}{m} \int_0^\infty du\, e^{\frac{-\epsilon^4 \Delta^2}{2m^2} u^2} = \frac{1}{2}. \tag{3.36}$$

Hence, the norm corresponding to the $|p| < \epsilon$ eigenstates corresponding to half the total norm for arbitrarily small ϵ, making it problematic to consider the remaining part as truly representative of the arrival time distribution. The reason that these eigenstates represent half the total norm can be understood by comparing Eq. 3.23 to the eigenstates of the unmodified operator given in Eq. 3.8. The form of the exponential in the unmodified eigenstates is $e^{(iET)}$, where $E = p^2/(2m)$ is the energy of a free particle. Hence for the eigenstates in Eq. 3.23, values of $|p| < \epsilon$ correspond to effectively negative energies [9, 19], so that excluding them excludes half the real line. This also helps us to understand why the modified operator is self-adjoint, since it effectively spans the real line, thus circumventing Pauli's objection.

3.3.2 Delgado and Muga

Delgado and Muga also consider a self-adjoint variation of the standard arrival time operator [3]. Delgado and Muga acknowledge from the beginning that their result is similar to a self-adjoint operator defined by Kijowski at the end of [8] based on the his axiomatically derived distribution, but they are unhappy with Kijowksi's derivation. Specifically, they believe the wave functions used by Kijowski in his mathematical derivation are not the standard wave functions of quantum mechanics, and hence they seek to derive a distribution in the standard framework of quantum mechanics. However, Kijowski himself rejects this criticism in [20], and insists that his self-adjoint operator exists in the usual Hilbert space of QM. For this reason, we may consider Delgado and Muga's derivation to be a modern restatement and expansion of Kijowski's [20, 21]. Their method of arriving at this operator is to first explicitly construct a new self-adjoint Hamiltonian which has a non-bounded spectrum i.e. with eigenvalues spanning the entire real line. This enables them to define an arrival time operator $\hat{T}$ via the commutation relation in Eq. 3.1. While the eigenstates of this operator cannot be identified directly with the physical time, Delgado and Muga describe how a physical interpretation can nonetheless be extracted from them.

To see how this works, we define projectors onto subspaces corresponding to positive and negative momentum plane waves

$$\theta(\alpha\hat{p}) = \int_0^\infty dp \ \left|\alpha p\right\rangle \left\langle\alpha p\right| . \tag{3.37}$$

We can now define a new self-adjoint Hamiltonian

$$\hat{\mathcal{H}} = (\theta(\hat{p}) - \theta(-\hat{p}))\hat{H} \tag{3.38}$$

where $\hat{H}$ is simply the regular plane wave Hamiltonian. Now consider the eigenstates of the free particle Hamiltonian, $\left|E, \alpha\right\rangle = (m/2E)^{1/4} \left|p = \alpha\sqrt{2mE}\right\rangle$, where $E \geq 0$.

Now if we relabel these as

$$|\varepsilon\rangle = \begin{cases} |+E\rangle \equiv |E,+\rangle\,, & \varepsilon \geq 0; \\ |-E\rangle \equiv |E,-\rangle\,, & \varepsilon \leq 0 \end{cases}$$

for $\varepsilon \in (-\infty, \infty)$, then we have that

$$\begin{gathered} \hat{\mathcal{H}}\,|E,\alpha\rangle = \alpha\,|E,\alpha\rangle\,; \\ \hat{\mathcal{H}}\,|\varepsilon\rangle = \varepsilon\,|\varepsilon\rangle\,. \end{gathered} \tag{3.39}$$

Hence we have defined a Hamiltonian with eigenvalues spanning the entire real line.

Now without explicitly defining an operator $\hat{T}$ in terms of $\hat{x}$ or $\hat{p}$, for example, we may nonetheless obtain such an operator simply by considering the states $|\varepsilon\rangle$. Then we may define the eigenstates of a $\hat{T}$ operator via

$$|\tau\rangle = \int_{\infty}^{\infty} d\varepsilon\; e^{i\varepsilon\tau}\,|\varepsilon\rangle\,. \tag{3.40}$$

Completeness and orthogonality of these eigenstates can be very easily demonstrated, suggesting that we can define a self-adjoint time operator as

$$\hat{T} = \int_{\infty}^{\infty} d\tau\;\tau\,|\tau\rangle\,\langle\tau|\,. \tag{3.41}$$

This operator automatically satisfies the correct commutation relation with our Hamiltonian due to the relationship between the eigenstates.

Delgado and Muga explain why this cannot be an operator corresponding to a physical time observable. We note a result from elementary quantum mechanics which states that the time reversal operator $\hat{\mathcal{R}}$ acts such that $\hat{\mathcal{R}}\,|p\rangle = |-p\rangle$. For a physical time, it should be the case that $|\tau\rangle \to |-\tau\rangle$. We may write

$$\langle p|\tau\rangle = (\frac{|p|}{2\pi m})^{\frac{1}{2}} \exp[i(\theta(p) - \theta(-p))\frac{p^2}{2m}\tau] = \langle -p|\tau\rangle^{\,*}. \tag{3.42}$$

Hence, applying the time reversal operator and inserting a resolution of the identity gives us

$$\hat{\mathcal{R}}\,|\tau\rangle = \int_{-\infty}^{\infty} dp\, \hat{\mathcal{R}}\,|p\rangle\,\langle p|\tau\rangle^* = \int_{-\infty}^{\infty} dp\ \ |-p\rangle\,\langle -p|\tau\rangle = |\tau\rangle\,. \tag{3.43}$$

The complex conjugate comes in due to the anti-unitarity of the time reversal operator. Hence, time reversal does not have the expected effect on $|\tau\rangle$. The covariance condition will also not hold since although $e^{-i\hat{H}t}\,|\tau\rangle \neq |\tau - t\rangle$, where $\hat{H}$ is the free particle propagator.

However, if we go back to Eq. 3.40, we see that we can decompose the eigenstates in the form

$$|\tau\rangle = |T = -\tau, -\rangle + |T = \tau, +\rangle \tag{3.44}$$

where

$$|T, \alpha\rangle \equiv \int_0^{\infty} dE\ e^{iET}\,|E, \alpha\rangle\,. \tag{3.45}$$

This decomposition means that it is now manifestly the case that

$$\hat{\mathcal{R}}\,|T, \alpha\rangle = |-T, -\alpha\rangle \tag{3.46}$$

so that each of these states could individually be associated with physical time. We also note that these states individually satisfy the covariance condition. Delgado and Muga now relate this to the probability current density defined in chapter 2. They assume that the quantum backflow effect is negligible when considering only positive momenta at large times, and in the presence of a perfect absorber as a detector (see section 4). Hence they assume that the average arrival time in the quantum case is a simple quantization of the arrival time at $x = 0$ in terms of the classical probability current:

$$\langle t\rangle_\alpha = \frac{\int_{-\infty}^{\infty} d\tau\ \tau\,\langle\psi_\alpha(\tau)|\,\hat{J}(0)\,|\psi_\alpha(\tau)\rangle}{\int_{-\infty}^{\infty} d\tau\ \ \langle\psi_\alpha(\tau)|\,\hat{J}(0)\,|\psi_\alpha(\tau)\rangle} \tag{3.47}$$

After much complicated manipulation, Delgado and Muga show that this can be expressed in the form

$$\langle t\rangle_\alpha = \alpha\,\langle\psi_\alpha|\,\hat{T}\,|\psi_\alpha\rangle \tag{3.48}$$

implying that the self-adjoint operator $\hat{T}$ defined in Eq. 3.41 measures the arrival time of particle at $x = 0$, such that in the semiclassical limit the form for the mean time of detection is of the appropriate form. Inserting Eq. 3.41 into Eq. 3.48, we obtain

$$\langle t \rangle_\alpha = \int_{-\infty}^{\infty} d\tau \; \alpha\tau |\, \langle \tau | \psi_\alpha \rangle \,|^2 \tag{3.49}$$

so that

$$\langle t \rangle_+ = \int_{-\infty}^{\infty} d\tau \; \tau |\, \langle T = \tau | \psi_+ \rangle \,|^2 \tag{3.50}$$

with a similar expression obtaining for negative momenta. We may then assume that

$$\Pi(T) = |\, \langle T = \tau | \psi_+ \rangle \,|^2 \tag{3.51}$$

for particles with positive momenta only. In fact, as we can see by looking at Eq. 3.42, this once again gives us Kijowski's distribution, but in this case we are limited to superpositions of positive or negative momentum states only due to the covariance condition given in section 2.

3.4 POVMs

In both of the examples above, one obtains a physically meaningful arrival time operator by restricting the domain $\mathcal{D}(\hat{T})$ over which the operator is to used. Grot et al. required us to restrict the domain to states with $|p| > \epsilon$. As we saw, this leads to a situation where an entire half of the norm is discarded when considering the arrival time probability, which may be unacceptable. Delgado and Muga chose to restrict the domain by only calculating arrival time distributions for states of purely positive or purely negative momentum. As we saw, despite limiting the domains of applicability, both of these methods obtained standards results for the arrival time distribution outlined in section 2. Despite this success, some authors believe these results to be problematic [9, 11]. If we are to obtain a mathematical theory of arrival times which has a definite connection to experimental reality, it seems artificial to limit the momentum states in the way prescribed in both of these cases, and it would be advantageous if a theoretical treatment could be devised which does not require limiting the domain in this way. One simple of way of doing this is simply by dropping the requirement for self-adjointness of the arrival time operator. Traditionally, it was assumed that one could not describe observables using non-self-adjoint operators, but many authors have pointed out that this need not necessarily be the case. Giannitrapani defines a *generalized observable* as a *positive operator valued measure* (POVM) on a measurable space, a POVM being defined as a map $\hat{B}$ from the real line to the algebra of positive operators (those giving only positive expectation values) on the Hilbert space of a quantum system [17]. A POVM satisfies the following conditions:

1. $\hat{B}(x) \geq \hat{B}(\emptyset), \qquad \forall x \in \mathbb{R};$
2. $\hat{B}(\bigcup x_i) = \sum \hat{B}(x_i)$, where x_i is a countable collection of disjoint elements on $\mathbb{R}$;
3. $\hat{B}(\mathbb{R}) = 1$.

As pointed out in [15], this set of conditions coincides with the requirements of quantum mechanics for describing probabilities related to observables. For the case that $\hat{B}(x)^2 = \hat{B}(x)$, this becomes a *projection valued measure* (PVM), which is the normal set of projectors onto orthogonal subspaces which we associate with self-adjoint operators. This is what we have generalized with this definition, and the above formulation suggests that projection onto *orthogonal* subspaces is not necessarily a requirement for something to be an observable. In a similar way to how we define probabilities for PVMs, $P([x_1, x_2]) = \mathrm{Tr}[\rho \hat{B}([x_1, x_2])]$ can be interpreted as the probability of measuring the observable in a state corresponding to the interval $[x_1, x_2]$. It can be shown, moreover, that a maximally symmetric operator has a unique POVM [15, 22], and as we saw above the Aharonov-Bohm time operator is maximally symmetric. Now recalling our eigenstates for this operator given in

Eq. 3.8, we can take the real line to correspond to the set of all times, and define

$$\hat{B}([T_1, T_2]) = \sum_\alpha \int_{T_1}^{T_2} dT \;\; |T, \alpha\rangle \langle T, \alpha| \,. \tag{3.52}$$

This is a POVM since it fulfils the three conditions laid out above: it's clearly positive; it is clearly additive for disjoint sets owing to the additivity property of integrals; and Eq. 3.10 tells us that $\hat{B}([-\infty, \infty]) = 1$. Hence the arrival time distribution for a state $|\psi\rangle$ is given by

$$\Pi(T) = \left\langle \psi \,\middle|\, \hat{B}(T) \,\middle|\, \psi \right\rangle = \sum_\alpha |\, \langle T, \alpha \,|\, \psi\rangle \,|^2. \tag{3.53}$$

Using the terms for $|T, \alpha\rangle$ given in Eq. 3.8 gives us the expression

$$\Pi(T) = \frac{1}{2\pi m} \left| \int_0^\infty dp \sqrt{p} e^{-iTp^2/(2m)} \psi(p) \right|^2 + \frac{1}{2\pi m} \left| \int_{-\infty}^0 dp \sqrt{-p} e^{-iTp^2/(2m)} \psi(p) \right|^2 \tag{3.54}$$

which is again none other than Kijowski's distribution. We also note that when we apply Eq. 3.9 to the expression for $\hat{B}(T)$ in terms of eigenstates, we get

$$e^{-i\hat{H}t} \hat{B}(T) e^{i\hat{H}t} = \hat{B}(T - t) \tag{3.55}$$

which means that according to Eq. 3.53

$$\Pi(T, \psi(0)) = \Pi(T - t, \psi(t)) \tag{3.56}$$

which again satisfies the covariance condition necessary for something to be considered a valid arrival time distribution.

3.5 Galapon's work on self-adjoint time operators

3.5.1 Refutation of Pauli's argument

Recent work by Galapon has contradicted assumptions made in previous work by suggesting that it is, in fact, possible to define self-adjoint operators which are conjugate to a semibounded Hamiltonian. To see how this is possible, we must expand our discussion in section 3.1.2 and look a little more closely at domains. In order to consider commutators of the form $\hat{a}\hat{b} - \hat{b}\hat{a}$ on a Hilbert space $\mathcal{H}$, we must recognize two facts.

Firstly, the domain of the commutator is not necessarily the same as the domain of either $\hat{a}$ or $\hat{b}$. The domain of $\hat{a}\hat{b}$, $\mathcal{D}_{ab}$, is the set of $|\psi\rangle \in \mathcal{H}$ such that $\hat{b}\,|\psi\rangle$ is in the domain of $\hat{a}$. Defining the domain of $\hat{b}\hat{a}$ similarly, the domain of the commutator is then simply $\mathcal{D}_{com} = \mathcal{D}_{ab} \cap \mathcal{D}_{ba}$.

Secondly, the domain on which the commutation relation $[\hat{a}, \hat{b}] = i$ is defined is not necessarily the whole domain $\mathcal{D}_{com}$. In the event that the canonical domain $\mathcal{D}_c$ on which the commutation relation is defined is not *dense* (i.e. if there is a vector other than the zero vector that is orthogonal to all elements of the domain), then the canonical domain is a subset of the commutator domain. We may then conceivably define a time operator conjugate to the Hamiltonian which is self-adjoint on the relevant domain $\mathcal{D}_c$.

To see why Pauli's theory is incomplete, we note that the argument for the impossibility of a self-adjoint operator conjugate to a semibounded Hamiltonian, given in section 3.1.1, relies upon Eq. 3.3. Clearly this relationship can naively be shown by induction to follow from Eq. 3.1, as long as no consideration is given to the domain of the operators. Galapon shows [23] that this relationship is not necessarily valid, hence invalidating Pauli's argument. We note that the domain of validity of the canonical conjugation relationship, $[\hat{H}, \hat{T}] = i$, is the specific canonical domain in this case $\mathcal{D}_c$, and that Pauli's arguments implicitly assume that we are dealing with the same domain throughout. Now consider the LHS of Eq. 3.3 with $n = 2$. We may write, for $|\psi\rangle \in \mathcal{D}_c$,

$$[\hat{H}, \hat{T}^2] |\psi\rangle = (\hat{H}\hat{T}^2 - \hat{T}^2\hat{H}) |\psi\rangle = (\hat{H}\hat{T} - \hat{T}\hat{H})\hat{T} |\psi\rangle + \hat{T}(\hat{H}\hat{T} - \hat{T}\hat{H}) |\psi\rangle . \quad (3.57)$$

Now for the second term on the very RHS, we obtain $i\hat{T} |\psi\rangle$, but for the first we do not in all cases. Note from the above discussion that the domain of $\hat{T}$ may actually be larger than the canonical domain. Galapon shows in [23] that one can consider cases where $\hat{T} |\psi\rangle \notin \mathcal{D}_c$ in general for $|\psi\rangle \in \mathcal{D}_c$. Therefore we can't apply the commutation relation to it, and $(\hat{H}\hat{T} - \hat{T}\hat{H})\hat{T} |\psi\rangle \neq i\hat{T} |\psi\rangle$ in general. So Eq. 3.3 is not true in general and Pauli's argument is rendered invalid.

3.5.2 Example of a self-adjoint time operator

Galapon shows how to construct a self-adjoint time operator canonically conjugate to a semibounded Hamiltonian by limiting the spatial domain to a region $x \in [-l, l]$ [24], so that we are now dealing with a Hilbert space $\mathcal{H}_l = L^2[-l, l]$. The position opeerator in this space is unique, and is the usual operator such that $\hat{x}\psi(x) = x\psi(x)$ for all $\psi(x)$ in the domain. The momentum operator is defined as $\hat{p}_\gamma$, such that $\hat{p}_\gamma\psi(x) = -i\partial\psi(x)/\partial x$, where the domain comprises those differentiable functions with square integrable first derivatives which satisfy the condition $\psi(-l) = e^{-2i\gamma}\psi(l)$ at the boundaries, where $|\gamma| < \pi/2$. Notice it is possible for there to be periodic boundary conditions, where $\gamma = 0$, and also non-periodic boundary conditions. We therefore have a range of momentum operators rather than just one. Defining the free Hamiltonian to be $\hat{H}_\gamma = \hat{p}_\gamma^2/2m$ as usual, $\hat{H}_\gamma$ and $\hat{p}_\gamma$ have the common eigenfunctions $\psi_k^\gamma(x) = \exp(i(\gamma + k\pi)x/l)/2l$, where $k = 0, \pm1, \pm2 \ldots$. Hence we have discrete spectra. Now, we define the time operator to be

$$\begin{aligned} \hat{T}_\gamma &= \frac{-m}{2}\left(\hat{x}\frac{1}{\hat{p}_\gamma} + \frac{1}{\hat{p}_\gamma}\hat{x}\right), \qquad \text{for } \gamma \neq 0; \\ \hat{T}_0 &= \frac{-m}{2}\left(\hat{x}\frac{1}{\hat{P}_0} + \frac{1}{\hat{P}_0}\hat{x}\right), \qquad \text{for } \gamma = 0. \end{aligned} \quad (3.58)$$

Here, $\hat{P}_0^{-1} = E\hat{p}_0^{-1}E$, where E is the projector onto the subspace orthogonal to the null space of $\hat{p}_0$. We can see that these are similar to the Aharonov-Bohm operator, but this time with $\hat{p}_\gamma$ momentum operators. We now cite without proof several facts about these operators, worked out both numerically and analytically by Galapon and collaborators in a series of papers [23–25]. Firstly, they are self-adjoint with discrete spectra. Secondly, each $\hat{T}_\gamma$ can be shown to be canonically conjugate to the corresponding Hamiltonian $\hat{H}_\gamma$. Thirdly, the eigenstates of the time operator evolve such that the expectation value of their position is at the origin at a time corresponding to their eigenvalues, and such that their width is a minimum at this time. Hence, it is perfectly reasonable to interpret these operators as time of arrival operators. Fourthly, in the limit that $l \to \infty$, the spectrum of the time operators becomes continuous, and the discrete time of arrival distribution given by the eigenstates of $\hat{T}_\gamma$ tends to Kijowski's distribution.

3.6 Criticism of attempts to define a time operator

Several authors express the point of view that the POVM method is the definitive expression of the arrival time concept in terms of operators [9, 11, 15, 17]. Mielnik and Torres-Vega, however, take issue with this view, and indeed with all attempts to define the arrival time using an operator approach [21]. In their view, most of the above attempts to define an arrival time operator are overly idealized, paying insufficient heed to the effect that actual measurement will have on an incoming particle. They detect an often unspoken assumption in Kijowski and all subsequent authors that there some very unintrusive detector which registers the arrival of particles. Either the detector is extremely weak, in which case there may be tunnelling across the detector which affects the incoming wave packet, or it is strong, which also affects the incoming wave packet. In neither case is there a free wave packet, which renders questionable any theory which assumes there is. Mielnik and Torres-Vega consider attempts to create self-adjoint time operators particularly unconvincing. They liken modifying the Hamiltonian, as done by Delgado and Muga, to representing the harmonic oscillator potential by $(\theta(\hat{x}) - \theta(-\hat{x}))\hat{x}^2/2$, which they consider to be an absurd proposition. For the case of a weak detector, Mielnik and Torres-Vega also question the validity of theories involving POVM measures. As we saw above, the probability distribution is a simple sum of the probabilities for arrival from the left or the right. However, it may also be possible that a single wave packet has coherent parts on both sides of a screen, in which case there will be interference terms not represented by the simple additive nature of this distribution. It seems then that in order to get a realistic picture of arrival times, it may be necessary to include detectors in the theory from the beginning. Such attempts are dealt with in the next chapter.

4 Detectors and Clocks

In this chapter we will look at so-called *operational* models for determining the arrival. This involves modelling the incoming particles in the presence of a detector or clock such that the Hamiltonian becomes $\hat{H} = \hat{H}_0 + \hat{H}_{apparatus}$. It is then a matter of solving the Schrödinger equation with this Hamiltonian so that an arrival time distribution may be extracted. We begin by revisiting Allcock, who provided one of the earliest attempts to model a detector with a complex potential. Then we will look at a more realistic detector model, where the arrival time of an atom to an area illuminated with a laser is associated with the time of first emission of a photon. Next we will see how using path integral calculations with a complex potential leads to realistic arrival time distributions. Finally, we shall describe quantum clocks, and look in detail at the recent use of a clock model to obtain arrival time distributions. We will find that the results often agree across different models, suggesting that they possess some universal validity.

4.1 Detectors

4.1.1 Allcock's complex potential

Following on from his argument, outlined in chapter 3, against the possibility of obtaining an arrival time distribution from a consideration of arrival time eigenstates, Allcock sought to understand if this goal could instead be accomplished by modelling incident waves interacting with a detector [6]. Allcock determined that a model detector, situated at $x = 0$, must remove probability as quickly as possible from the area $x > 0$ for incident waves. This is consistent with an irreversible process which we would associate with a physical detector, whereby the incoming wave probability is quickly diminished due to its transformation into something which might register as a signal. His proposal was to model this process by modifying the Hamiltonian, introducing a potential $-iV\theta(x)$ so that the Schrödinger equation becomes

$$i\frac{\partial\psi}{\partial t}+\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}+iV\theta(x)=0. \tag{4.1}$$

Defining the total norm to be $N=\langle\psi(T)|\psi(T\rangle$, we can calculate the probability of detection in terms of the rate at which the norm is being absorbed - i.e. $\Pi(T)=-dN(T)/dT$. We can then use Eq. 4.1 to show that

$$\Pi(T)=-\frac{dN}{dT}=2V\int_0^\infty dx\ |\psi(x,T)|^2. \tag{4.2}$$

Probability is disposed of within a time $\delta T\sim(2V)^{-1}$, and Allcock interprets this to mean that one can only determine the time of arrival to within this uncertainty.

Allcock now solves the Schrödinger equation by matching boundary conditions at $x=0$. The amplitudes of transmitted and reflected waves in the energy representation, where $\psi(E)$ is the incident wave, are given by

$$\phi_{tr}(E)=\frac{2}{1+E^{-1/2}(E+iV)^{1/2}}\psi(E); \tag{4.3}$$

$$\phi_{ref}(E)=\frac{1-E^{-1/2}(E+iV)^{1/2}}{1+E^{-1/2}(E+iV)^{1/2}}\psi(E). \tag{4.4}$$

Using a version of Eq. 3.13 along with Eq. 4.2, we obtain

$$\begin{aligned}\Pi(T)=&\frac{1}{2\pi}\int_{-\infty}^{\infty}dE\int_{-\infty}^{\infty}dE'\left[\frac{2\psi(E)e^{iET}}{E^{1/4}(1+E^{-1/2}(E+iV)^{1/2})}\right]^*\times\\&\times\frac{2\psi(E')e^{iE'T}}{E'^{1/4}(1+E'^{-1/2}(E'+iV)^{1/2})}\frac{iV((E'+iV)^{1/2}+(E+iV)^{1/2*})}{2\pi(E'-E+2iV)}.\end{aligned} \tag{4.5}$$

The problem here, of course, is that $\Pi(T)$ depends on V, which ought not to be case. Allcock identifies two possibilities for obtaining a V-independent distribution: the limits $V\to0$ and $V\to\infty$. The latter case produces a very small time uncertainty, but a perusal of the transmission and reflection amplitudes shows that this limit corresponds to very little transmission of incident probability, leading to failure of the measurement. Allcock consequently rejects this option. The low V limit corresponds to a higher time uncertainty. Allcock assumes that the above $\Pi(T)$ is actually a convolution of the ideal arrival time distribution Π_{id} with some V-dependent response function of the detector:

$$\Pi(T) = \int dT \; R(V, T - T')\Pi_{id}(T'). \tag{4.6}$$

Making suitable approximations for low V, $\Pi_{id}(T)$ can be extracted from Eq. 4.5 as

$$\Pi_{id}(T) = \frac{1}{2\pi m} \left| \int_0^\infty dp \sqrt{p} e^{-iTp^2/(2m)} \psi(p) \right|^2 = \langle \psi(T) | \frac{1}{m} (\hat{p}^{\frac{1}{2}} \delta(\hat{x}) \hat{p}^{\frac{1}{2}}) | \psi(T) \rangle \,, \tag{4.7}$$

which is yet again Kijowski's distribution, for the case of particles travelling only to the right.

In the background of this derivation, however, is the increased time uncertainty due to the fact that the probability is not absorbed sufficiently fast by the potential. Allcock considers this to be a general result, and expresses the view that one cannot simultaneously have perfect detection efficiency and perfect time resolution. As we have seen, increasing the potential to speed up absorption results in increased reflection. Muga, Brouard and Macias [26] contest Allcock's claim, by suggesting a model for a perfect absorbing potential which absorbs the incoming wave function in an arbitrarily short time. For a limited range of momenta, they suggest that the arrival time distribution could be thus obtained to arbitrary accuracy. As will see below Aharonov et al. also suggest a model for a perfect detector [27]. However, when they subsequently attach the system to a clock, a time uncertainty is introduced.

Allcock's claim that one cannot obtain an arrival time distribution in the limit $V \to \infty$ is challenged by several authors, who believe that a realistic distribution can be obtained by normalizing the small amount of probability which is absorbed in this limit. We will discuss this further below.

4.1.2 Scattering methods

In recent times there has been some interest in studying the arrival time problem with a more realistic detector model. A recent attempt has been made to describe how the arrival time at $x = 0$ of an atom travelling to the right can be measured by illuminating a region $x \geq 0$ with a laser [10, 28–30]. Its arrival time in this case will be taken as the time at which the first photon is emitted from the atom within the illuminated region. To model this, we make use of results from quantum optics. If we think of the incoming atom as being a two-level system, then the Hamiltonian for the motion through the laser field of the atoms for which no photon has been detected is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{\Omega\theta(\hat{x})}{2}(|1\rangle\langle 2| + |2\rangle\langle 1|) - \frac{i}{2}\gamma\,|2\rangle\langle 2| \tag{4.8}$$

Here, $|1\rangle$ and $|2\rangle$ are the ground state and excited state of the atom respectively, Ω is the *Rabi frequency*, which parametrizes the strength of the coupling between levels, and γ is the inverse lifetime of the state $|2\rangle$. Again, we simply have to solve the Schrödinger equation for this Hamiltonian in the regions $x \geq 0$ and $x < 0$, and match conditions at $x = 0$. This will enable us to obtain an arrival time distribution as the negative time derivative of the norm as outlined in section 4.1.1. As there is effectively a step potential at $x = 0$, this will cause both a transmitted and reflected component of the wave function. There will also be solution for both the $|1\rangle$ and $|2\rangle$ states. The calculations are quite involved, but one ultimately obtains

$$\Pi(T) = \gamma \int_{-\infty}^{\infty} dx\, |\psi^{(2)}(x,t)|^2, \tag{4.9}$$

where $\psi^{(2)}(x,t)$ is the wave function of the excited state. (Note that $\psi^{(2)}(x,t)$ is a function of both γ and Ω.) Hence the first photon emission distribution is the decay rate multiplied by the probability of finding the atom in the excited state. There are two reasons why this distribution differs from the ideal case - reflection of the wave packet and delay in detection.

This is, of course, similar to what pertains for Allcock's complex potential. The equivalent of modifying the potential in that case if modifying the decay rate γ and the Rabi frequency Ω. The solution for the wave functions is dependent on both of these parameters. One finds that the effect of sending $\gamma \to \infty$, while keeping Ω fixed at a constant value, is to send the reflection coefficient to zero, but also to decrease the photon absorption by the atom, and hence increase the detection delay. If, on the other hand, one keeps γ/Ω constant, sending $\gamma \to \infty$ results in total reflection of the wave packet [28]. Hence we have the same dilemma as with Allcock's example.

The solution is via the same type of deconvolution considered by Allcock. They ansatz is made that the measured distribution is a convolution such that $\Pi = W * \Pi_{id}$, where $W(T)$ gives the probability density for the detection of the first photon emitted by an atom at rest when driven by a laser. The procedure is fundamentally similar to Allcock's. Taking the limit $\gamma \to \infty$ with Ω constant is just like sending $V \to 0$ in Allcock's example, and the resulting distribution is deconvoluted to obtain an ideal distribution. The result presented in [28] is

$$\Pi_{id}(T) = \langle \psi(T) | \frac{1}{2m} (\hat{p}\delta(\hat{x}) + \delta(\hat{x})\hat{p}) | \psi(T) \rangle \tag{4.10}$$

which is, of course, the probability current density of Eq. 2.6.

In fact, as demonstrated in [31] and [32], in the large γ limit, we may manipulate Eq. 4.8 so that the equation to be solved can be thought of as effectively a standard one-channel Schrödinger equation with a complex potential given by $V = -i\Omega^2/(2\gamma)$. For a region of length L illuminated with a laser, the Hamiltonian is then given by $\hat{H} = \hat{p}^2/(2m) - iV$, with $V = 0$ outside of this region. The detection rate is then given by

$$\Pi(T) = 2V \int_0^L dx \, |\psi(x,t)|^2. \tag{4.11}$$

Muga et al. [30] solve this in the $V \to \infty$ limit, where there is a great deal of reflection due to the high potential. Defining p_0 to be the initial average momentum, they hence normalize the distribution to give

$$\Pi_N(T) = \frac{\Pi(T)}{\int dT\, \Pi(T)} = \frac{1}{mp_0} \langle \psi(T) | \hat{p}\delta(\hat{x})\hat{p} | \psi(T) \rangle \,, \tag{4.12}$$

which is a new form for the arrival time distribution. In fact, this is $(2/p_0)\langle\hat{\tau}\rangle_T$, where $\hat{\tau}$ is called the *kinetic energy density* operator. This operator is clearly not obtainable by reordering operators as with $\hat{\Pi}_J$ and $\hat{\Pi}_K$ in eqs. 2.10 and 2.11.

We have assumed that $\gamma \to \infty$ to obtain this result, but it is experimentally difficult to adjust the decay rate. However, Muga et al. demonstrate that even for γ fixed, where we can no longer approximate the illuminated region as a complex potential, it is possible to solve the full two-level problem and still obtain $(2/p_0)\langle\hat{\tau}\rangle_T$ by deconvolution [30].

4.1.3 Pulsed measurements

An interesting argument is provided by Echanobe et al. to demonstrate that the result obtained for the arrival time distribution in Eq. 4.12 is also obtained if one considers the pulsed detection in intervals of δT, where $\delta T \to 0$ corresponds to the condition $V \to \infty$ [33]. They note that the propagator for pulsed measurements, which periodically absorb the wave function by 'kicking' it with an imaginary potential, is given by $\hat{U}_k(0, \delta T) = \exp(-i\hat{H}_0 \delta T)\exp(-i\hat{V}\delta T)$, whereas the operator for the continuous potential is given by

$$\begin{aligned} \hat{U}(0, \delta T) &= \exp(-i(\hat{(H)}_0 + \hat{V})\delta T) \\ &= \exp(-i\hat{H}_0 \delta T)\exp(-i\hat{V}\delta T) + \mathcal{O}(\delta t^2[\hat{V}, \hat{H}_0]). \end{aligned} \tag{4.13}$$

As we saw, this propagator gives the distribution of Eq. 4.12 when normalized. Numerical studies show that in the limit where $\delta T \to 0$ and $V \to \infty$, the last term on the RHS of Eq. 4.13 goes to zero, so that the distribution achieved by the pulsed method is the same as that in Eq. 4.12. Since pulsed measurements at very small intervals give rise to the Zeno effect, this result not only shows that reflection due to high continuous potentials is an expression of the Zeno effect, but that this effect is not enough to stop us from obtaining a reasonable arrival time distribution. This result also justifies an original argument by Allcock in [6] that a complex potential was an adequate approximation to a pulsed measurement detector model.

4.1.4 Path integral approach with a complex potential

A recent approach to calculating the arrival time distribution in the presence of a complex potential has involved employing path integrals in order to find expressions for propagators in terms of $\delta(\hat{x})$ and $\hat{p}$, thus enabling comparison with the results obtained in chapter 2 and in subsequent chapters [34,35]. We shall here assume that the particles are travelling to the left and that the potential is located in $x < 0$. We begin by observing that the arrival time distribution in the presence of a somewhat general potential $V(x) = -iV_0\theta(-x)f(x)$ can be expressed as

$$\Pi(T) = 2\langle\psi(T)|V(\hat{x})|\psi(T)\rangle = 2V_0\int_{-\infty}^{0} dx\ f(x)\,|\langle x|\exp(-i\hat{H}T|\psi\rangle|^2 \tag{4.14}$$

We now wish to express this in a form involving $\delta(\hat{x})$ and $\hat{p}$. In order to do this, we make use of a theorem about propagators called the Path Decomposition Expansion (PDX), a proof of which can be found in [36]. Recall that we may use a path integral approach to derive a propagator via

$$g(x_1, T|x_0, 0) = \langle x_1|\hat{U}|x_0\rangle = \int \mathcal{D}x\ e^{iS} \tag{4.15}$$

where S is the action, and the integral is a weighted sum over the infinity of paths from the spacetime points x_0 and $T = 0$ to x_1 at time T. According to the PDX composition law, if we take an intermediate surface in spacetime between the start and end points, we may express this sum over paths in terms of the restricted sum over paths g_r to the intermediate surface which never cross the surface, and the sum over paths from the intermediate surface to the end point, which is given by an unrestricted propagator. Taking the intermediate surface in our case to be the arrival point $x = 0$, we may express this composition law mathematically as

$$g(x_1, T|x_0, 0) = \frac{i}{2m} \int_0^T dt \; g(x_1, T|0, t) \frac{\partial g_r}{\partial x}(x, t|x_0, 0)|_{x=0}. \tag{4.16}$$

Hence since our propagator is, from Eq. 4.14,

$$g(x_1, T|x_0, t) = \langle x_1 | \exp(-i\hat{H}(T-t)) | x_0 \rangle \,, \tag{4.17}$$

where $\hat{H} = \hat{H}_0 - iV_0\theta(-x)f(x)$, we may express it in an alternative form using Eq. 4.16. It may be shown the method of images [37] that

$$\frac{\partial g_r}{\partial x}(x, t|x_0, 0)|_{x=0} = 2\frac{\partial g}{\partial x}(0, t|x_0, 0)\theta(x_0) \tag{4.18}$$

so that we may easily calculate that

$$g(x_1, T|x_0, t) = -\frac{1}{m} \int_0^T dt \;\; \langle x_1 | \exp(-i\hat{H}(T-t)\delta(\hat{x})\hat{p}\exp(-i\hat{H}_0 t|x_0\rangle \,. \tag{4.19}$$

We may now insert this into Eq. 4.14, rearrange, and change variables from t and t' to $s = T - t$ and $s' = T - t'$ to obtain

$$\Pi(T) = \frac{2V_0}{m^2} \int_0^T ds \int_0^T ds' \int_{-\infty}^0 dx f(x) \langle 0 | exp(i\hat{H}^\dagger s') | x \rangle \langle x | \exp(-i\hat{H}s) | 0 \rangle \times$$
$$\times \langle \psi | \exp(i\hat{H}_0(T-s'))\hat{p}\delta(\hat{x})\hat{p}\exp(-i\hat{H}_0(T-s)) | \psi \rangle \,. \tag{4.20}$$

The $\hat{p}\delta(\hat{x})\hat{p}$ structure emerges from the observation that $\delta(\hat{x})\hat{A}\delta(\hat{x}) = \delta(\hat{x}) \langle 0|\hat{A}|0\rangle$. We already notice the structure of the last term is similar to that obtained via the scattering method above. We may now take either of the two limits suggested by Allcock in order to obtain a V_0-independent term, namely $V_0 \to \infty$ and $V_0 \to 0$. In the first case, the relationship $\delta T \sim (V_0)^{-1}$ suggests near perfect resolution, whereas in the latter case we would expect to obtain an ideal distribution by deconvolution as suggested by Eq. 4.6. Let us therefore take each limit, motivated by the observation in [30] and [33] that distributions can be obtained in the high V limit despite

Allcock's objections.

Starting with the limit $V_0 \to \infty$, we note that due to the V_0 in the $\hat{H}$ term, the integrals over s and s' will be concentrated around $s = 0$ and $s' = 0$. Hence we may estimate that

$$\Pi(T) = C \langle\psi| \exp(i\hat{H}_0 T)\hat{p}\delta(\hat{x})\hat{p} \exp(-i\hat{H}_0 T)|\psi\rangle \tag{4.21}$$

where C is constant due to the large V_0 value. It is now possible by a further application of the path integral approach to calculate C exactly, so that the final expression is

$$\begin{aligned}\Pi(T) =& \frac{2}{m^{3/2}V_0^{1/2}} \langle\psi| \exp(i\hat{H}_0 T)\hat{p}\delta(\hat{x})\hat{p} \exp(-i\hat{H}_0 T)|\psi\rangle \\ =& \frac{2}{m^{3/2}V_0^{1/2}} \langle\psi(T)|\hat{p}\delta(\hat{x})\hat{p}|\psi(T)\rangle .\end{aligned} \tag{4.22}$$

This coincides with the value obtained in the $V_0 \to \infty$ limit obtained via the scattering method in section 4.1.2 above.

In the limit that $V_0 \to 0$, we may make the approximation that

$$\langle x| \exp(-i\hat{H}s)|0\rangle \approx \langle x| \exp(-i\hat{H}_0 s)|0\rangle \exp(-V_0 s) = \left(\frac{m}{2\pi i s}\right)^{1/2} \exp\left(\frac{imx^2}{2s} V_0 s\right). \tag{4.23}$$

The resulting integral is complicated but achievable, as long as we assume that $V_0 T \gg 1$, and the result is

$$\Pi(T) = 2V_0 \int_0^T dt \exp(-2V_0(T-t)) \frac{1}{2m} \langle\psi(t)|(\hat{p}\delta(\hat{x})\Sigma(\hat{p}) + \Sigma^\dagger(\hat{p})\delta(\hat{x})\hat{p})|\psi(t)\rangle , \tag{4.24}$$

where $\Sigma(\hat{p}) = \hat{p}/(2m(\hat{H}_0 + iV_0))^{1/2}$. In the low V_0 limit, this reduces to $\hat{p}/|\hat{p}|$, which is the momentum sign function - i.e. -1 for particles travelling to the left. We notice that the resulting distribution conforms to the form for a convolution given in Eq. 4.6, and hence we obtain

$$\Pi_{id}(T) = \frac{-1}{2m}(\hat{p}\delta(\hat{x}) + \delta(\hat{x})\hat{p}), \tag{4.25}$$

which is of course the same as the probability current density given in Eq. 2.10, up to a sign convention. An interesting point here is that we seem to have derived a probability distribution with a detector model which may be negative for certain values due to the backflow effect. This may be due to some of the approximations made when taking the $V_0 \to 0$ limit. We may still reasonably take this as an *approximate* probability distribution [35].

4.1.5 Other detector methods

There have of course been many other attempts to obtain arrival time distributions using model detectors. Halliwell [38] explains that a realistic detector model must include a certain irreversibility. Standard QM, where evolution is modelled by unitary operators, is fundamentally reversible. If we model a detector as a two level system where detection implies a transition from one level to another, then it is possible that the transition could reverse itself. Since realistic detectors are effectively irreversible, Halliwell models a two level detector (with levels $|0\rangle$ and $|1\rangle$) in the presence of an environment, such that

$$\hat{H} = \hat{H}_0 + \hat{H}_d + \hat{H}_\epsilon + V(x)\hat{H}_{d\epsilon}, \tag{4.26}$$

where the RHS is the Hamiltonian of the particle, detector and environment, and the interaction between the particle and environment respectively. Modelling the environment as a collection of harmonic oscillators, and using the master equation for the time-evolution of this system yields, for a pure state

$$|\Psi(T)\rangle = \exp(-i\hat{H}_0 T - \gamma V T/2)\, |\Psi(0)\rangle\,, \tag{4.27}$$

where γ is some parameter. This term is derived from a consideration of the entire system above, and justifies the previous use of complex potentials to model detectors. In this context, probabilities for detection in an interval $[0, T]$ are derived. Hence, this method mirrors the modelling of detectors with a complex potential, but has the added benefit of including a realistic irreversibility in the system.

In the context of the fluorescence model described in section 4.1.2, another method was introduced called *operator normalization.* We saw in Eq. 4.12 that in the presence of reflection, one can normalize the distribution in order to get

$$\Pi_N(T) = \frac{\Pi(T)}{\int dT\, \Pi(T)}. \tag{4.28}$$

Another way to normalize, according to [10], is on the level of the $\hat{\Pi}$ operators whose expectation values give arrival time distributions. Let us define a detection probability operator $\hat{B}$ such that its expectation value over incoming states is $\langle\psi_{in}|\hat{B}|\psi_{in}\rangle$ = total probability for detection. Then we may normalize the operator $\hat{\Pi}$ whose expectation value gives the unnormalized distribution ($\Pi(T)$ in Eq. 4.28) to get $\hat{\Pi}^{ON} = \hat{B}^{-1/2}\hat{\Pi}\hat{B}^{1/2}$, so that the expectation value of this quantity automatically gives us a normalized distribution. This corresponds in an experimental situation to passing the incoming waves through an initial potential in order to appropriately filter them. The upshot of this method is that one may mathematically show that in the limit $V \to \infty$, one obtains Kijowski's distribution.

4.2 Clocks

In this section we will summarize the approach to the arrival time problem using quantum clocks and, state the features of two examples [27] considered by Aharonov et al. Then we will describe in detail a particular calculation of arrival time distributions using a model clock given in [39]. A clock is defined by Peres as a "system which passes through a succession of states at constant time intervals" [40]. The idea behind a quantum clock is that it is coupled to whatever system we are interested in, and the interaction causes the clock to keep a record of some dynamical time variable of the system in question. The Hamiltonian of the system becomes $\hat{H} = \hat{H}_0 + \hat{H}_{clock}$ with a certain region, and another variable y is introduced to record the state of the clock. If the clock Hamiltonian turns off at the arrival point, the idea is that we may solve the Schrödinger equation, and since the clock stops evolving we may use its final state in the limit $T \to \infty$ to obtain a time-independent distribution $\Pi(y)$. This will give us the arrival time. Aharonov et al. [27] looked at several toy models of clocks in an attempt to understand the uncertainties involved in obtaining an arrival time distribution. They begin by looking at simple linear model where $\hat{H}_{clock} = \theta(-\hat{x})\hat{p}_y$, where y is the clock variable whose evolution can be associated to the movement of a pointer on the clock. Aharonov et al. solve the Schrödinger equation with this Hamiltonian and demonstrate that the peak of the distribution $y(k)$ for a wave packet coincides in the classical limit with the classical arrival time. They discover, however, a fundamental limitation on the accuracy of the clock given by $\langle E \rangle \Delta T > 1$, where $\langle E \rangle$ is the average energy of an incoming wave packet, due to the occurrence reflection.

Another system they describe models a clock and a detector together, where the detector triggers the clock to turn off once the particle is detected. This trigger is given as a two level spin system, where $|\uparrow_z\rangle$ corresponds to the clock being on and $|\downarrow_z\rangle$ to the clock being off. The Hamiltonian of the particle and trigger is given by $\hat{H}_{trigger} = \hat{H}_0 + \alpha(1 + \sigma_x)\delta(\hat{x})$. In the limit $\alpha \to \infty$, the repulsive Delta function cause total reflection of the incoming wave if the spin state is $|\uparrow_x\rangle$ and has no effect if it is $|\uparrow_x\rangle$. Given the relationship between $|\uparrow_x\rangle / |\downarrow_x\rangle$ and $|\uparrow_z\rangle / |\downarrow_z\rangle$ from elementary QM, we may write the evolution of a system of an initial wave state $|\psi\rangle$ with an 'on' clock $|\uparrow_z\rangle$ in terms of transmitted and reflected wave functions as

$$|\psi\rangle |\uparrow_z\rangle \to \frac{1}{2} |\uparrow_z\rangle (|\psi_{ref}\rangle + |\psi_{tr}\rangle) + \frac{1}{2} |\downarrow_z\rangle (|\psi_{ref}\rangle - |\psi_{tr}\rangle). \tag{4.29}$$

Hence the detector turns the clock off with probability 1/2. It is then possible to include an arbitrary number of detectors N, which means that the probability of at least one of the N spins flipping becomes $1 - 2^{-N}$. Hence the detector may be made arbitrarily accurate. Aharonov et al. use this model to refute Allcock's claim that one cannot absorb a particle in an arbitrarily short time. They now couple this system to a clock given by $\hat{H}_{clock} = (1+\sigma_z)\hat{p}_y$. They again show that an uncertainty arises due to the clock. Hence while there may be perfect detectors, one must then couple them to a clock which reintroduces an uncertainty.

We now look in detail at a model clock studied in [39] by Yearsley et al.

4.2.1 Clocks with path integral methods

Yearsley et al. use a clock model in conjunction with the type of path integral techniques we saw in section 4.1.4, in an attempt to extract a reasonable arrival time distribution [39]. They begin by considering a Hamiltonian for the combined system of a particle and model clock: $\hat{H} = \hat{H}_0 + \lambda\theta(\hat{x})\hat{H}_{clock}$. The last term describes the interaction between the clock and the particle. We can now consider this coupling in two regimes: the weak case, where $E \gg \lambda\epsilon$, E being the characteristic kinetic energy of the particle; and the strong case, where $E \ll \lambda\epsilon$. We may think of these as being similar to the conditions $V \to 0$ and $V \to \infty$ respectively for the complex detector models given above. As we can see, for a particle in $x > 0$ travelling to the left, this interaction continues until the particle reaches the arrival point, at which point the clock stops and the time is recorded by the position of the clock pointer variable. To obtain an arrival time distribution, let us begin by assuming that the clock Hamiltonian may be written as

$$\hat{H}_c = \int d\epsilon \ \left|\epsilon\right\rangle \left\langle\epsilon\right|, \tag{4.30}$$

where the $|\epsilon\rangle$ are eigenvalues of the clock Hamiltonian which form an orthonormal basis. We now take the pointer variable of the clock to be y, and the total system of clock and particle to be initial in the state $|\Psi_0\rangle = |\psi_0\rangle\,|\phi_0\rangle$, with the kets on the RHS representing the particle and the clock respectively. We may then get the total wave function as a function of time by calculating

$$\begin{aligned}\Psi(x,y,T) &= \langle x|\,\langle y|\exp(-i\hat{H}T\,|\psi_0\rangle\,|\phi_0\rangle \\ &= \int d\epsilon\,\langle y|\epsilon\rangle\,\langle\epsilon|\phi_0\rangle\,\langle x|\exp(-i\hat{H}_0 T i\lambda\theta(\hat{x})\epsilon T)|\psi_0\rangle\end{aligned} \tag{4.31}$$

This will enable us to calculate a probability distribution for y in the limit that $T \to \infty$, and since the value of y should be frozen in place at the arrival time, we can obtain an arrival time distribution. Such a distribution would be given by

$$\Pi(y) = \int_{-\infty}^{\infty} dx\, |\Psi(x, y, T)|^2. \tag{4.32}$$

We now wish to solve for the propagator using the same kind of path integral techniques we saw previously. Previously, we assumed that the potential entered the Schrödinger equation *after* a particle entered the region $x < 0$ and sought to solve for the propagator accordingly. This time, we assume that the extra term in the Hamiltonian enters the Schrödinger equation only in the period *before* the crossing. Hence if we consider our Hamiltonian to be of the form $\hat{H} = \hat{H} + V_0\theta(\hat{x})$, we find that the same calculation that led to Eq. 4.19 now gives us

$$\begin{aligned} g(x_1, T|x_0, t) &= \langle x_1| \exp(-i\hat{H}T|x_0\rangle \\ &= \frac{1}{m}\int_0^T dt\; \langle x_1| \exp(-i\hat{H}(T-t)\delta(\hat{x})\hat{p} \exp(-i(\hat{H}_0 + V)t|x_0\rangle\,. \end{aligned} \tag{4.33}$$

In this case, contrary to Eq. 4.19, we note that we still have the V in the second exponential due to the differing θ-function employed in this case.

Now let us make the assumption that $V = \lambda\epsilon \ll E$ in order to consider the weak coupling case. Let us look at Eq. 4.31, and note that we can use Eq. 4.33 to obtain

$$\begin{aligned} \Psi(x, y, T) = \int d\epsilon\, \langle y|\epsilon\rangle\, \langle\epsilon|\phi_0\rangle \int_0^T dT \times \\ \times\, \langle x| \exp(-i(\hat{H}_0 + \lambda\epsilon\theta(\hat{x}))(T-t)\delta(\hat{x})\hat{p} \exp(-i\hat{H}_0 + \lambda\epsilon t)|\psi_0\rangle \end{aligned} \tag{4.34}$$

Now we can use a simplifying assumption due to the weak regime. Recalling that $\delta(\hat{x}) = |0\rangle\langle 0|$, we note that we have in this equation the term $\langle x| \exp(-i(\hat{H}_0 + \lambda\epsilon\theta(\hat{x}))(T-t)|0\rangle$. This is a propagator from $x = 0$ to $x > 0$. Although this propagator is a sum over all possible paths, including those which cross the point $x = 0$ many times, the small $\lambda\epsilon$ assumption allows us to assume most of the contribution will come from more direct propagation, so that we may neglect the effect of the potential-type term. Hence $\langle x|\exp(-i\hat{H}(T-t)|0\rangle \approx \langle x|\exp(-i\hat{H}_0(T-t)|0\rangle$.

We can now calculate the distribution $\Pi(y)$ defined in Eq. 4.32. Using similar techniques to those in section 4.1.4, the distribution can be calculated to be

$$\Pi(y) = \frac{1}{m^2}\int d\epsilon\, d\epsilon' \,\langle\phi_0|\epsilon'\rangle\,\langle\epsilon|\phi_0\rangle \int_0^T dt\, dt' \times \\ \times \langle\psi_0|\exp(i(\hat{H}_0+\lambda\epsilon')t')\hat{p}\delta(\hat{x})\hat{p}\exp(-i(H_0+\lambda\epsilon)t)|\psi_0\rangle\,\langle 0|\exp(-i\hat{H}_0(t'-t)|0\rangle\,. \tag{4.35}$$

The result of this expression now relies crucially on taking $T\to\infty$ in the integration limit. These integrals can then be done to obtain the form

$$\Pi(y) = \int_0^\infty dt\; |\Phi(y,t)|^2 (\frac{-1}{2m})\,\langle\psi(t)|\delta(\hat{x})\hat{p}+\hat{p}\delta(\hat{x})|\psi(t)\rangle \tag{4.36}$$

which once again gives us the convolution form of Eq. 4.6. The Φ term depends on the clock wave function, and the remainder is of course the probability current density once again. Hence we have used a clock with weak coupling to obtain the same result as in the low V limit for the detector model above.

Naturally, we may now seek to obtain the strong coupling form of $\Pi(y)$. The calculation is again rather involved, but suffice it to say that for the case $\hat{H}_c = \hat{p}_y$, we can take suitable approximations and the limit $T\to\infty$ in order to obtain

$$A\,\langle\psi_0|\exp(i\hat{H}_0 y/\lambda)\hat{p}\delta(\hat{x})\hat{p}\exp(-i\hat{H}_0 y/\lambda)|\psi_0\rangle\,, \tag{4.37}$$

where A is a constant. This again, of course, gives us the kinetic energy density with $t = y/\lambda$. It is also shown in [39] that this applies for more general clock models.

4.3 Summary of detectors and clocks

We now emphasize the fact that we have obtained similar results for a variety of methods. A recurring theme first noted by Allcock is that arrival time distributions can most suitably be obtained in limits $V\to\infty$ and $V\to 0$ or equivalent. When we send $V \to 0$ in the complex potential model, which corresponds to the low emission probability limit in the fluorescence model and the weak coupling regime in the clock model, we obtain the expectation value of the probability current for the arrival time distribution. Allcock claimed that the $V \to \infty$ regime should be rejected due to excessive wave function reflection at the potential boundary, but we have seen that subsequent methods have normalized the small transmitted current in order to obtain results proportional to the kinetic energy density. It seems, then, that many methods have challenged the suggestion that an ideal distribution cannot be obtained due to uncertainty effects.

The fact that different detector models and arbitrary clock models lead to the same results in these limits suggest that it is reasonable to deduce that these results are generically true. We also note that in one limit we obtain the probability current as our probability density. The fact that this is often negative justifies the increasing interest shown in the backflow effect in recent years.

5 Decoherent histories

The decoherent histories approach considers a closed quantum system in terms of a set of histories of the system, and attempts to assign probabilities to the different histories. However, if we wish to know the position of a particle at a series of times so that we can define its history, we may think of it as being similar to setting up a series of double slits to measure the particle's position at the each time step [1]. This helps us to understand why there will, in fact, be interference between different histories. The result is that one cannot simply add up probabilities for each history to obtain a total probability of 1. There are, however, some situations in which one might group together sets of histories - called 'coarse graining' - so that we achieve disjoint and additive probabilities for distinct groups of histories. We may then treat them as normal probabilities and hence define a probability distribution. In this case, the histories are said to be 'decoherent'. The applicability of this approach to the time of arrival problem is obvious, and is explored below.

We shall begin by taking a brief look at one of the first such attempts to derive a time of arrival distribution due to Yamada and Takagi. We shall discover that they reach a negative conclusion regarding the possibility of defining a time of arrival distribution. We will see that this issue has recently been solved by a consideration of the Zeno effect. A recent derivation of a time of arrival distribution using decoherent histories is then presented in detail.

5.1 A brief account of Yamada and Takagi's decoherent histories approach

An early attempt to investigate the time of arrival concept via the decoherent histories approach was that of Yamada and Takagi [41–43]. They use path integral techniques in order to investigate the propagation of a particle from the 1+1 dimensional spacetime position $A = (X_A, T_A)$ to $B = (X_B, T_B)$. They described the motion between these two points with a propagator defined as a familiar sum over paths $\Phi(B;A) = \sum_{A\to B} e^{iS(XT)}$, which we may think of a sum of histories of the system going from A to B. For a wave function, the evolution is determined by the equation

$$\psi(B) = \int_{-\infty}^{\infty} dX_A \Phi(B;A)\psi(A). \tag{5.1}$$

In accordance with the standard rules of QM, we may observe that at over a surface of constant time T_B, the normalization condition will hold for space wave functions:

$$\int_{-\infty}^{\infty} dX_B \, |\psi(B)|^2 = \int_{-\infty}^{\infty} dX_B \left| \int_{-\infty}^{\infty} dX_A \; \Phi(B;A)\psi(A) \right|^2 \tag{5.2}$$

Let us now consider an general intermediate surface $\mathcal{S}$ between A and B - it may be constant T or constant X or a mixture of both which is divided into different section of length Δl. We wish to consider only a contribution to the propagator which is a sum over paths travelling through the surface an odd number of times through a certain sequence of subsections Δl_n. It may cross over and back many times. We now wish to assign a probability of the set of histories given by this particle sequence of crossings so that

$$P(\Delta l_n, \mathcal{S}) = \int_{-\infty}^{\infty} dX_B \left| dX_A \int_{\Delta l_n} d\lambda \; \Phi_n(B;\lambda,\mathcal{S},A)\psi(A) \right|^2, \tag{5.3}$$

such that

$$\int_{\Delta l_n} d\lambda \; \Phi_n(B;\lambda,\mathcal{S},A) = \sum_{A\to\Delta l_n\to B} e^{iS[X(T)]}. \tag{5.4}$$

Two conditions are now given such that these probabilities can be additive and disjoint as required:

$$\Phi(B;A) = \sum_n \sum_{\Delta l_n} \int_{\Delta l_n} d\lambda \; \Phi_n(B;\lambda,\mathcal{S},A); \tag{5.5}$$

$$Re \int dX_B \int dX_A \int dX'_A \; \Phi^*_m(D;\lambda;A)\Phi_n(B;\lambda';A')\psi^*(A)\psi(A') \propto \delta_{mn}\delta(\lambda-\lambda'). \tag{5.6}$$

The latter condition is clearly the condition for decoherence between histories.

When the intermediate surface $\mathcal{S}$ is taken to be a constant time surface, Yamada and Takagi demonstrate that one can define sets of propagators which satisfy these conditions. This is consistent with standard QM where a distribution for a wave function is obviously definable at constant time. The question which we are interested in is that of a constant x surface at $x = 0$. The propagators in this case are calculated by a random walk method since we are dealing with time intervals rather than space intervals on $\mathcal{S}$. After much mathematical manipulation, they deduce that only for a special case is it possible to define the probability of a particle arriving for the first time at $x = 0$ within a finite time interval. This is for a wave function antisymmetric about $x = 0$, and in this case the probability of crossing is zero. It goes without saying that this result is not physical.

Subsequent attempts were made to address this problem. For example, Halliwell and Zafiris [44] sought to couple the system to an environment made of harmonic oscillators in order to induce decoherence. They claimed to induce decoherence of states by this method, which had a sensible classical limit. However, this claim was later shown to be incorrect (see footnote in [35]).

Recent work has suggested the reason for the failure of Yamada and Takagi to produce sensible results via the decoherent histories method [35, 45]. As we saw above, Yamada and Takagi use path integrals to obtain the propagators $\Phi(B, A)$. In particular they use a method similar to the path decomposition expansion described in section 4.1.4, in which the total propagator is expressed in terms of a composition law involving a restricted propagator, which describes propagation on one side of $x = 0$ without every crossing over. Halliwell aad Yearsley discovered a fundamental problem with such path integrals: if we think of the derivation of path integrals in terms of time slices, the condition that the width of the time slices goes to zero corresponds to a continuous monitoring of the system. This means that restricted propagators in $x > 0$ fall foul of the Zeno effect, which as we saw in section 2.4 causes continuously monitored states within a Hilbert space to remain within that Hilbert space. A naive application of path integrals in the decoherent histories approach can therefore lead to unphysical results. In particular, this explains why the only case producing a probability for crossing $x = 0$ in Yamada and Takagi's analysis produces a zero probability. The results of Halliwell and Zafiris also suffer from an inadequate appreciation of the Zeno effect. We will now see how recent work has been done which takes account of this effect, and consequently achieves a sensible arrival time distribution.

5.2 Halliwell and Yearsley's model

5.2.1 The projector formalism for decoherent histories

Halliwell and Yearsley [35] were the first to demonstrate that imposing the condition of decoherence recovers standard values for the arrival time distribution seen in chapter 4. In their approach, the projector formalism is used to obtain an arrival time distribution. We therefore review its basic principles [1, 35, 46].

We can characterize a quantum mechanical history by a class operator, C_α, which is made up of a string of projectors P onto the various possibilities at each time instant. Hence we may write

$$C_\alpha = P_{\alpha_n} e^{-i\hat{H}(t_n - t_{n-1})} P_{\alpha_{n-1}} \dots P_{\alpha_1}. \tag{5.7}$$

We see that the class operator has the structure of a projector onto a subspace followed by standard unitary time evolution across a finite time step followed by another projection etc. between the initial and final times. We can now clearly see from the sum rules of orthogonal projectors that

$$\sum_\alpha C_\alpha = e^{-i\hat{H}t}, \tag{5.8}$$

where t is the total time interval. Each history can be assigned a probability via

$$p(\alpha_1, \alpha_2, \dots) = p(\alpha) = Tr(C_\alpha \rho C_\alpha^\dagger), \tag{5.9}$$

but these histories need not necessarily obey the additivity property of probabilities due to the quantum interference of the histories. We may quantify the interference between histories in term of the decoherence functional

$$D(\alpha, \alpha') = Tr(C_\alpha \rho C_{\alpha'}^\dagger). \tag{5.10}$$

We want this quantity to go to zero in order for there to be decoherence, and the consequent possibility of defining disjoint and additive probabilities. We may also define the so-called *quasi-probability* which is

$$q(\alpha) = Tr(C_\alpha \rho e^{i\hat{H}t}). \tag{5.11}$$

Note that we may thus show that

$$q(\alpha) = p(\alpha) + \sum_{\alpha \neq \alpha'} D(\alpha, \alpha'). \tag{5.12}$$

So when there is decoherence between histories, we have that $q(\alpha) = p(\alpha)$. Unlike for a regular probability, in the absence of decoherence it's possible for $q(\alpha)$ to be negative. These facts will be useful later on.

5.2.2 The Zeno effect and decoherent histories

In order to consider the arrival time problem, we consider a situation where a particle is in $x > 0$ travelling leftwards. We my define two class operators C_c and C_{nc} corresponding to histories which cross $x = 0$ within the time interval [0,t], and those which do not respectively. Halliwell and Yearsley now note a serious problem with previous definitions of class operators, if the time interval ϵ between projections is taken to zero. In the limit $\epsilon \to 0$, the class operators fall foul of the Zeno effect, as described in section 5.1 above, since this limit of continuous projection onto the positive x-axis corresponds to continuous measurement of the particle. Indeed, one may calculate that the probability for not crossing obeys $p_{nc} \to 1$ in this limit.

The solution suggested in this case is to recall the result of Echanobe et al., presented in section 4.1.3 above, which demonstrated that constant measurement at intervals ϵ within a region may be approximated as a continuous potential throughout the region. We may therefore define the $C_{nc} = \exp(-i\hat{H}_0 t - V_0\theta(-\hat{x})t)$, as in [33]. Now comparing with Eq. 5.8, we obtain the relation

$$e^{-i\hat{H}_0 t} = C_c(t) + C_{nc}(t). \tag{5.13}$$

Using this relation with our definition of $C_{nc}(t)$ gives us

$$C_c(t) = e^{-i\hat{H}_0 t} - e^{-i\hat{H}_0 t - V_0\theta(-\hat{x})t}. \tag{5.14}$$

5.2.3 Infinitesimal class operators

Now let us divide the interval $[0, t]$ into n steps of length ϵ. By Eq. 5.13 we have $e^{-i\hat{H}\epsilon} = C_c(\epsilon) + C_{nc}(\epsilon)$. We may now obtain a useful and instructive expression for $e^{-i\hat{H}t}$ by noting that:

$$\begin{aligned} e^{-i\hat{H}t} &= (e^{-i\hat{H}\epsilon})^n \\ &= (e^{-i\hat{H}_0\epsilon})^{n-1} C_{nc}(\epsilon) + (e^{-i\hat{H}_0(t-\epsilon)})C_c(\epsilon). \end{aligned} \tag{5.15}$$

We may repeatedly substitute for $e^{-i\hat{H}\epsilon}$ to obtain

$$e^{-i\hat{H}_0 t} = C_{nc}(t) + \sum_{k=0}^{n-1} e^{-i\hat{H}_0(t-(k+1)\epsilon)} C_c(\epsilon) C_{nc}(\epsilon). \tag{5.16}$$

Now, taking the infinitesimal ϵ limit, we obtain via suitable substitution and approximation for the class operators

$$e^{-i\hat{H}_0 t} = C_{nc}(t) + \int_0^t dt'\, e^{-i\hat{H}_0(t-t')} V e^{-i\hat{H}_0 t - V_0\theta(-\hat{x})t}. \tag{5.17}$$

Hence for any time interval, we may define the crossing class intervals

$$C_c^k(t) = \int_{t_k}^{t_{k+1}} dt'\ e^{-i\hat{H}_0(t-t')} V e^{-i\hat{H}_0 t - V_0\theta(-\hat{x})t} = \int_{t_k}^{t_{k+1}} dt'\ C_c(t'). \tag{5.18}$$

5.2.4 Obtaining an arrival time distribution

Let us make the assumption that all energy scales $E \gg V_0$, and attempt to derive an arrival time distribution. We now recall the definition of probabilities in section 5.2.1. Applied to our coarse-grained crossing class operators in Eq. 5.18, we obtain that the probability of crossing in a time interval $[t_k, t_{k+1}]$ is $p(t_k, t_{k+1}) = Tr(C_c^k \rho (C_c^k)^\dagger)$ and the quasi-probability is $q(t_k, t_{k+1}) = Tr(C_c^k \rho e^{i\hat{H}t})$. Now consider the quantity

$$\langle x| e^{i\hat{H}_0 t} C_c(t) |\psi\rangle = V_0 \langle x| e^{i\hat{H}_0 t} \theta(-\hat{x}) e^{-i\hat{H}_0 t - V_0\theta(-\hat{x})t)} |\psi\rangle\,. \tag{5.19}$$

This may be calculated using the same path integral techniques which we saw in section 4.1.4 leading to Eq. 4.25, assuming that the value of V_0 is very small. The result is that we may express the coarse-grained crossing operator as

$$e^{i\hat{H}_0 t} C_c^k = \int_{t_k}^{t_{k+1}} dt'\ \frac{-1}{2m}(\hat{p}\delta(\hat{x}) + \delta(\hat{x})\hat{p}). \tag{5.20}$$

If we then calculate the *quasi*-probability using this expression, we have

$$q(t_k, t_{k+1}) = Tr(C_c^k \rho e^{i\hat{H}t}) = \int_{t_k}^{t_{k+1}} dt'\ \frac{-1}{2m} \langle\psi| (\hat{p}\delta(\hat{x}) + \delta(\hat{x})\hat{p}) |\psi\rangle\,, \tag{5.21}$$

which gives us our probability as an integral in terms of the probability density current $\hat{\Pi}_J$. This is, of course the same result we obtained for many of the detector methods we examined in the $V_0 \to 0$ regime. But this is a quasi-probability, and as Eq. 5.12 tells us this will only be equal to an actual (positive) probability in the case that there is decoherence. This seems to give us some insight into the backflow effect, since it suggests that the negativity of current will only lead to negative values when there is no decoherence - i.e. when we have not sufficiently coarse-grained the probabilities by integrating over a time integral in order to induce decoherence. The only thing that remains is to show that there *is* decoherence between histories. In fact we must show that two types of decoherence are satisfied:

$$\begin{aligned} D_{kk'} &= Tr(C_c^k \rho (C_c^{k'})^\dagger) = 0; \\ D_{k,nc} &= Tr(C_c^k \rho (C_{nc})^\dagger) = 0. \end{aligned} \tag{5.22}$$

Here, k and k' represent different coarse-grained sets of histories. For a model Gaussian wave packet, for example, Halliwell and Yearsley show that as long as the time interval within which one measures arrival time does not approach the *Zeno time*, then decoherence is satisfied. Zeno time is defined as $m\sigma/|p_0|$, where σ is the wave packet width and p_0 is its momentum. Hence there is again a time uncertainty associated with arrival time measurement.

6 Conclusion

In the foregoing, we have examined many of the main strands in recent research into the time of arrival concept in quantum mechanics. It is clear that this is a hugely active and complicated area of research, of which we were only able to give a summary.

In chapter 2 we saw how some commonly occurring candidates for the arrival time distribution are derived. The probability density current was suggested by a simple quantization of the classical arrival time probability for an ensemble, and Kijowski's distribution was derived by considering a group of axioms limiting the form that an arrival time distribution might reasonably take. In subsequent chapters, both of these candidates recur frequently, suggesting both are plausible arrival time distributions.

In chapter 3, we looked at many different attempts to define an arrival time operator canonically conjugate to the Hamiltonian. We found that, for many authors, the semibounded nature of the Hamiltonian was considered an impediment to creating a straightforward time of arrival operator with continuous eigenvalues across the entire real line. The most often adduced candidate for an arrival time operator, the Aharonov-Bohm operator, was produced by a symmetrization of the classical term for arrival time. It was found that this operator is non-self-adjoint, requiring more sophisticated methods to obtain an arrival time distribution. We saw how some authors attempted to modify the Aharonov-Bohm operator to make it self-adjoint. However, this involved excluding certain momentum states from the domain of the operator. The theory of POVMs sought to solve the issue by noting that a self-adjoint operator is not necessary in order to define an arrival time distribution. It is sufficient that it is maximally symmetric. We also looked at a refutation of Pauli's argument against the possibility of defining suitable self-adjoint time operators. We saw how a self-adjoint operator with a discrete spectrum could be defined in a spatially confined domain, and that taking the size of this domain to infinity resulted in convergence of the discrete arrival time distribution to Kijowski's distribution.

Regardless of the method used, it is obvious that defining a suitable arrival time operator is highly non-trivial, and it may suggest the need to rethink or expand upon the standard measurement theory of QM.

In chapter 4, we looked at several models of systems containing detectors and clocks. We saw how Allcock was the first to suggest that a detector be modelled with an absorbing complex potential, and that this idea has recurred ever since. A more realistic detector model was described whereby the arrival time was obtained as the time of emission of the first photon by an atom entering a laser field. A method for using path integral techniques with a complex potential to obtain arrival time distributions was also examined. And the use of a model clock to obtain similar distribution was also described. In each case, we saw that in the limit of high potential or strong coupling, we obtained a term proportional to the kinetic energy density for the arrival time distribution, and in the opposite limit, we obtained the probability current density. The remarkable frequency with which these distributions recur suggests that they are generically true, and the fact that there are different distributions for different regimes points to the difficulty of separating an ideal distribution from the process by which it is measured. It will be interesting to see what future research says about the role of the Zeno effect in the different results for different regimes.

In chapter 5, we looked at the decoherent histories approach to defining arrival times. We saw that early research using path integrals arrived at unphysical conclusions about the measurement of arrival times. We saw, however, that the most successful application of the decoherent histories idea to arrival times solved this problem by paying attention to the Zeno effect. Again the probability current density was obtained in a certain limit, giving further evidence for the validity of this quantity as an arrival time distribution.

We note that a recurring theme in these investigations has been an uncertainty associated with the measurement of arrival time. In the detector models, for example, we saw that the ideal distribution had to be obtained via a deconvolution from the actual measured distribution due to time delay. Aharonov et al. showed for several clock models that there was a time uncertainty associated with the measurement in each case. And in the decoherent histories approach, we saw that the Zeno effect meant that an uncertainty had to be introduced into arrival time measurement. Indeed, the Zeno effect is an ongoing area of active research. It seems that it will inevitably come into any discussion involving the use of path integrals or detectors, and it is vital that it is fully understood for a wide variety of scenarios. In addition, the backflow effect and the issue of negative probabilities is a vital area of study given the recurrence of the probability current density throughout the literature.

Although we have focused on arrival time in this work, it is important to recognize that it is not the only dynamical time variable of interest. For example, dwell time or transit time is the time that a particle takes to traverse a particular region. The question of how long a particle takes to tunnel through a barrier is an area of hugely active research. The widely-accepted form for the dwell time operator is a self-adjoint operator which commutes with the Hamiltonian. This avoids many of the issues we see with the arrival time, and yet there are many complications relating to this problem which could themselves fill a lengthy review. Suffice it to say that there are many facets of time in quantum mechanics and each poses its particular challenges.

Finally, we note that the conditions under which arrival time was generally examined in this review were very restrictive. Consideration of any problem in one dimension should be a precursor to generalizing it to three dimensions. Furthermore, we have generally dealt with free particles, without much consideration for the types of complicated potentials which occur in real systems. Some of the methods outlined, such as Galapon's work on operators, admit of straightforward generalizations to scenarios with such potentials, while others, such as Kijowski's derivation, do not. Finally, it will be necessary to generalize this work to the relativistic regime in order for it to be of universal application. Although some work has been done in this area already, it will undoubtedly be a major focus of future research.

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